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# CONTENTS.

Page

On the Vibration of a Gas enclosed between a Rigid and Fixed Spheroid and a Concentric Sphere vibrating with a prescribed normal velocity: Hrishikesh Sircar ... 1—10

Application of Quaternion Analysis to find the astatic centre of co-planar forces: Bibhutibhusan Sen ... ... 11-12

On a certain limit connected with pairs of integers: S. D. Chowla ... ... ... ... 13—15

On the Periodic Solutions of the Differential Equation $d^2y/dx^2 + (A + 16q \cos 2nx)y = 0$: Rama Shankar Varma ... ... ... 16 - 30

On an Expression for a Class of Integral Functions: P. L. Srivastava ... ... ... 31-32

NOTES AND QUESTIONS:

Bhaskara and Samclishta Kuttaka: A. A. Krishnaswami Ayyangar ... ... *1—7*

Generalisation of Some Properties of the Pedal Line: G. A. Srinivasan ... ... ... *8—11*

Half-circle of Curvature of a Conic: S. Purushotham ... ... ... *12—14*

Solutions ... ... ... *15—22*

Questions for Solution ... ... ... *23-24*

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# ON THE VIBRATION OF A GAS ENCLOSED BETWEEN A RIGID AND FIXED SPHEROID AND A CONCENTRIC SPHERE VIBRATING WITH A PRESCRIBED NORMAL VELOCITY

By Hrishikesh Sircar,

*Lecturer, University of Dacca*

The problem of the scattering of sound-waves by an obstacle when the source producing the waves and the nature of the obstacle are known, received considerable attention from distinguished physicists and mathematicians like Clebsch,* Rayleigh † Stokes ‡ and others. Dr S. K. Bannerjee § has considered a problem in which the source producing the disturbance is a spherical surface with a prescribed vibration, the obstacle being a rigid and fixed sphere. Dr. N. M. Basu and myself || have recently considered the scattering of sound-waves due to prescribed vibrations of a cylindrical surface in the presence of a rigid and a fixed cylindrical obstacle. In the present paper I have attempted to consider a case in which a spherical surface is vibrating with a prescribed velocity within or without a concentric, fixed and rigid spheroid of small eccentricity. In the following analysis, I have supposed that the vibrating sphere is within the spheroid; the other case when it is without it may be similarly treated by taking the appropriate wave-functions to represent the diverging and converging waves. Further, the spheroid has been supposed to be prolate but it may be replaced by an oblate one with the introduction of necessary changes.

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* 'Über die Reflexion an einer kugelfläche : *Jour. fur. Math.* Bd, 61, p 63 (1863).

† Investigation of the disturbance produced by a spherical obstacle on the waves of sound : *Proc. Lond. Math. Soc.*, Vol. 4, (1872).

‡ *Mathematical and Physical Papers*, Vol. 4, p. 314.

§ " On Sound-waves ... ... :" *Bull. Cal. Math. Soc.*, Vol. 4, (1912-13).

|| *Bulletin Cal. Math. Soc*, Vol XVIII. No. 2, p. 87.

## SECTION 1.

Let the common centre be chosen as the origin, the axis of revolution of the spheroid as the $z$-axis. Let $(\zeta, \mu_1, \omega)$ denote the prolate spheroidal co-ordinates of a point in space, of which the polar and Cartesian co-ordinates are respectively $(r\ \theta, \omega)$ and $(x, y, z)$. Then, we have

$$x = k_0 \sqrt{\zeta^2 - 1}\ \sqrt{1 - \mu_1^2}\ \cos\omega = r \sin\theta \cos\omega$$

$$y = k_0 \sqrt{\zeta^2 - 1}\ \sqrt{1 - \mu_1^2}\ \sin\omega = r \sin\theta \sin\omega$$

$$z = k_0\, \mu_1\, \zeta = r \cos\theta,$$

so that $$r^2 = k_0^2\, (\zeta^2 + \mu_1^2 - 1)$$

and $$\tan\theta = \frac{\sqrt{\zeta^2 - 1}\ \sqrt{1 - \mu_1^2}}{\zeta\, \mu_1} = \frac{\sqrt{1 - \mu^2}}{\mu} \qquad (\mu = \cos\theta),$$

where $$k_0 = a'e' = a\, e, \qquad \ldots \qquad \ldots \quad (\alpha)$$

$a', e'$ = the semi-major axis and eccentricity respectively, of the generating ellipse of the spheroid passing through the point under consideration,

$a, e$ = the semi-major axis and eccentricity respectively of the generating ellipse of the given spheroid.

Therefore, $$\frac{\partial r}{\partial \zeta} = k_0^2\, \frac{\zeta}{r}, \text{ and } \frac{\partial \mu}{\partial \zeta} = \frac{\mu\,(1 - \mu^2)}{\zeta\,(1 - \zeta^2)} \qquad \ldots \quad (1)$$

Let the given spheroid be represented by

$$r = a\,[1 + \varepsilon\, P_2\,(\mu)],$$

where $$a = a_0\,(1 - \varepsilon) \text{ and } \varepsilon = \frac{e^2}{3} \qquad \ldots \quad (\beta)$$

## SECTION 2.

Assuming that the sphere*, perfectly isolated, is placed in an unlimited medium and that the vibration of the sphere is symmetrical with

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* A complete solution of the problem was given by Sir George Stokes in 1868 "On the communications of vibrations from a vibrating body to a surrounding gas." *Phil Trans.*, 1868. Also *Math. and Phys. Papers*., Vol. IV, p. 299.

195231

respect to the axis of revolution of the spheroid and is represented by

$$\sum_{n=0}^{m} A_n P_n(\mu) e^{ikct},$$

in which the A's are known constants and $m$ is finite, the velocity-potential of the un-obstructed diverging wave-system is given by

$$\phi = \sum_{n=0}^{m} c_n (kr)^n f_n(kr) P_n(\mu) . e^{ikct},$$

where

$$c_n = -\frac{A_n}{\frac{\partial}{\partial b}[(kb)^n f_n(kb)]},$$

$b$ denoting the radius of the vibrating sphere, the function $f$ for the representation of a diverging wave-system being that introduced by Professor Lamb for such a purpose in his Hydro-dynamics.*

The functions $\psi$ † and $f$ represent the series

$$\psi_n(\zeta) = \frac{1}{1.3.5 \dots (2n+1)}\left[1 - \frac{\zeta^2}{2(2n+3)} + \frac{\zeta^4}{2.4(2n+3)(2n+5)} \dots\right]$$

and

$$f_n(\zeta) = \frac{i^n e^{-i\zeta}}{\zeta^{n+1}}\left[1 + \frac{n(n+1)}{2i\zeta} + \frac{(n-1)n(n+1)(n+2)}{2.4.(i\zeta)^2} + \dots + \frac{1.2 \dots 2n}{2.4 \dots 2n\,(i\zeta)^n}\right]$$

The abovementioned functions satisfy recurrence formulae of the types

$$\left.\begin{aligned} \psi'_n(\zeta) &= -\zeta\,\psi_{n+1}(\zeta) \\ \text{and} \quad \psi_{n-1}(\zeta) &= \zeta\,\psi'_n(\zeta) + (2n+1)\,\psi_n(\zeta) \end{aligned}\right\} \quad \dots \quad (2)$$

## SECTION 3.

Let $\phi$ denote the velocity potential of the motion within the contemplated space; then besides satisfying the wave-equation, it must

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* Lamb: *Hydro-dynamics*, 5th ed. p. 479–80.

† See: Watson, *Theory of Bessel Functions*: p. 56.

also satisfy the two boundary conditions

$$\frac{\partial \phi}{\partial r} = - \sum_{n=0}^{m} A_n P_n(\mu) e^{ikct}, \quad \text{on } r = b, \quad \dots \quad (1)$$

and

$$\frac{\partial \phi}{\partial n} = 0, \quad \text{on } r = a\,[1 + \varepsilon P_2(\mu)], \quad \dots \quad (2)$$

Assume

$$\phi = \sum_{n=0}^{m} P_n(\mu)(kr)^n [R_n f_n(kr) + S_n \psi_n(kr)] e^{ikct}$$

$$+ \varepsilon \sum_{n=m+1}^{n=\infty} P_n(\mu)(kr)^n [R'_n f_n(kr) + S'_n \psi_n(kr)] e^{ikct}$$

where the R's, R′'s S's and S′'s are unknown constants to be determined from the above boundary conditions.

From the first boundary condition, we have

$$\sum_{n=0}^{n=m} A_n P_n(\mu) = k \sum_{n=0}^{n=m} (kb)^{n-1} P_n(\mu) [n \{R_n f_n(kb) + S_n \psi_n(kb)\}$$

$$+ kb \{R_n f'_n(kb) + S_n \psi'_n(kb)\}]$$

$$+ k\varepsilon \sum_{n=m+1}^{n=\alpha} (kb)^{n-1} P_n(\mu) [n \{R'_n f_n(kb) + S'_n \psi_n(kb)\}$$

$$+ kb \{R'_n f'_n(kb) + S'_n \psi'_n(kb)\}].$$

Hence, equating the co-efficients of the various zonal harmonics on both sides, we have,

$$\frac{A_n}{k(kb)^{n-1}} = R_n [n f_n(kb) + kb f'_n(kb)] + S_n [n \psi_n(kb) + kb \psi'_n(kb)],$$

$$\dots \quad (b)$$

where to $n$ we must assign all values from $o$ to $m$,

and

$$0 = R'_n [n f_n(kb) + kb f'_n(kb)] + S'_n [n \psi_n(kb) + kb \psi'_n(kb)],$$

where $n$ may have all values from $m + 1$ to infinity.

## SECTION 4.

Again, from the second boundary condition, since the element $dn$ of normal at $(x, y, z)$ on the surface $\zeta$ = constant is given by

$$dn = \frac{d\zeta}{h}, \text{ where } \frac{1}{h^2} = \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2,$$

we have

$$0 = \frac{\partial \phi}{\partial \zeta} = \frac{\partial \phi}{\partial r} \cdot \frac{\partial r}{\partial \zeta} + \frac{\partial \phi}{\partial \mu} \cdot \frac{\partial \mu}{\partial \zeta}, \text{ on } r = a\,[1 + \varepsilon P_2(\mu)]$$

$$= \frac{k^2\zeta}{r} \frac{\partial \phi}{\partial r} + \frac{\mu(1-\mu^2)}{\zeta(1-\zeta^2)} \frac{\partial \phi}{\partial \mu}, \text{ on } r = a\,[1 + \varepsilon P_2(\mu)],$$

(from (1), Section 1

$$= \frac{k^2\zeta}{r} \frac{\partial}{\partial r} \Big[ \sum_{n=0}^{m} P_n(\mu)(kr)^n \{ R_n f_n(kr) + S_n \psi_n(kr) \}$$

$$+ \varepsilon \sum_{n=m+1}^{\infty} P_n(\mu)(kr)^n \{ R'_n f_n(kr) + S'_n \psi_n(kr) \} \Big]$$

$$+ \frac{\mu(1-\mu^2)}{\zeta(1-\zeta^2)} \frac{\partial}{\partial \mu} \Big[ \text{expression in the above square brackets} \Big]$$

on the given spheroid

$$= \frac{(kk_0)^2}{e} \Big[ \sum_{n=0}^{m} P_n(\mu)(kr)^{n-2} R_n \{ n f_n(kr) + kr f'_n(kr) \}$$

$$+ \sum_{n=0}^{m} S_n P_n(\mu)(kr)^{n-2} \{ n r_n(kr) + kr \psi'_n(kr) \}$$

$$+ \varepsilon \sum_{m=m+1}^{\infty} R'_n P_n(\mu)(kr)^{n-2} \{ n f_n(kr) + kr f'_n(k\gamma) \}$$

$$+ \varepsilon \sum_{n=m+1}^{\infty} S'_n P_n(\mu)(kr)^{n-2} \{ n \psi_n(kr) + (kr \psi'_n(kr) \} \Big]$$

$$+ \frac{e^3}{e^2-1} \mu(1-\mu^2) \sum_{n=0}^{m} P'_n(\mu)(kr)^n \{ R_n f_n(kr) + S_n \psi_n(kr) \}$$

$$+ \varepsilon \sum_{n=m+1}^{\infty} P'_n(\mu)(kr)_n \{ R'_n f_n(kr) + S'_n \psi_n(kr) \} \Big],$$

since $\zeta = \frac{1}{e}$, on the spheroid.

∴ On the given spheroid,

Or $$0 = (ka)^2 \Big[ \sum_{n=0}^{m} R_n . P_n(\mu)(kr)^{n-2} \{ nf_n(kr) - (kr)^2 f_{n+1}(kr) \}$$

$$+ \varepsilon \sum_{n=m+1}^{\alpha} R'_n \times \{ \text{the same expression as above} \}$$

$$+ \sum_{n=0}^{m} S_n \times \{ \text{a similar expression with } f \text{ replaced by } \psi \}$$

$$+ \varepsilon \sum_{n=m+1}^{\alpha} S'_n \times \{ \text{the same expression as in the third line} \}$$

$$- 3\varepsilon \sum_{n=0}^{m} \mu (1-\mu^2) P'_n(\mu)(kr)^n \{ R_n f_n(kr) + S_n \psi_n(kr) \},$$

neglecting higher powers of $e$ beyond $e^2$ as far as possible, and using the relations ($\alpha$) and ($\beta$) of section 1, and the recurrence-formula (11) of section 2.

$$= \sum_{n=0}^{m} R_n . P_n(\mu)(ka)^n \Big[ n \{ f_n(ka) - (n-2)\varepsilon(1-P_2) f_n(ka)$$

$$+ \varepsilon (ka)^2 (1-P_2) f_{n+1}(ka) \} - (ka)^2 \{ f_{n+1}(ka)$$

$$- n\varepsilon(1-P_2) f_{n+1}(ka) + \varepsilon (ka)^2 (1-P_2) f_{n+2}(ka) \} \Big]$$

$$+ \sum_{n=0}^{m} S_n \times [\text{same expression as above with } f \text{ replaced by } \psi]$$

$$+ \varepsilon \sum_{n=m+1}^{\alpha} R'_n . P_n(\mu)(ka)^n [ n f_n(ka) - (ka)^2 f_{n+1}(ka) ]$$

$$+ \varepsilon \sum_{n=m+1}^{\alpha} S'_n \times [\text{same expression as in above line with } f \text{ replaced by } \psi]$$

$$- 3\varepsilon \sum_{n=0}^{m} \mu (1-\mu^2) P'_n(\mu)(ka)^n [ R_n f_n(ka) + S_n \psi_n(ka) ],$$

on expanding by Taylor's theorem, the terms involving $\varepsilon$ in the last three square brackets being ignored, due to the presence of the factor $\varepsilon$.

$$= \sum_{n=0}^{\infty} R_n . (ka)^n \, [P_n(\mu) \, \{ n f_n(ka_0) - n(n-2) \varepsilon f_n(ka_0)$$

$$- (ka_0)^3 f_{n+1}(ka_0) + 2n\varepsilon (ka_0)^2 f_{n+1}(ka_0) - \varepsilon (ka_0)^4 f_{n+2}(ka_0) \}$$

$$+ \varepsilon \{ n(n-2) f_n(ka_0) - 2n (ka_0)^2 f_{n+1}(ka_0) + (ka_0)^4 f_{n+2}(ka_0) \}$$

$$\times \left\{ \frac{3}{2} \cdot \frac{(n+1)(n+2)}{(2n+3)(2n+1)} P_{n+2}(\mu) + \frac{n(n+1)}{(2n+3)(2n-1)} P_n(\mu) \right.$$

$$\left. + \frac{3}{2} \cdot \frac{n(n-1)}{(2n+1)(2n-1)} P_{n-2}(\mu) \right\} \Big]$$

$$+ \sum_{n=0}^{\infty} S_n \times \text{[an expression similar to the above with } f \text{ replaced by } \psi]$$

$$+ \varepsilon \sum_{n=m+1}^{\infty} R'_n \, P_n(\mu) (ka_0)^n \, [n f_n(ka_0) - (ka_0)^2 f_{n+1}(ka_0)]$$

$$+ \varepsilon \sum_{n=m+1}^{\infty} S'_n \times \text{[a similar expression as with } R'_n \text{ in which } f \text{ is replaced by } \psi]$$

$$+ 3\varepsilon \sum_{n=m+1}^{\infty} \frac{n(n+1)}{2n+1} (ka)^n \, [R_n f_n(ka_0) + S_n \psi_n(ka_0)]$$

$$\times \left[ \frac{n+2}{2n+3} P_{n+2}(\mu) - \frac{2n+1}{(2n+3)(2n-1)} P_n(\mu) \right.$$

$$\left. - \frac{n-1}{2n-1} P_{n-2}(\mu) \right]$$

since

$$(1-\mu^2) \frac{d}{d\mu} P_n(\mu)$$

$$= n \, [P_{n-1}(\mu) - \mu \, P_{n+1}(\mu)]^*$$

$$= n \left[ P_{n-1}(\mu) - \frac{(n+1) P_{n+1}(\mu) + n P_{n-1}(\mu)}{2n+1} \right]^*$$

$$= \frac{n(n+1)}{2n+1} \left[ P_{n-1}(\mu) - P_{n+1}(\mu) \right]$$

* Whittaker and Watson: ***Modern Analysis***, 3rd. ed., pp. 308-309.

Therefore,

$$\mu\,(1-\mu^2)\frac{d}{d\mu}\,P_n(\mu)$$

$$=\frac{n\,(n+1)}{2n+1}\left[\frac{n\,P_n(\mu)+(n-1)\,P_{n-2}(\mu)}{2n-1}-\frac{(n+2)\,P_{n+2}(\mu)+(n+1)\,P_n(\mu)}{2n+3}\right]$$

$$=\frac{n\,(n+1)}{2n+1}\left[\frac{n-1}{2n-1}\,P_{n-1}(\mu)-\frac{n+2}{2n+3}\,P_{n+2}(\mu)+\frac{2n+1}{(2n-1)(2n+3)}\,P_n(\mu)\right],$$

and

$$P_2(\mu)\,P_n(\mu)=\frac{3}{2}\,\frac{(n+1)(n+2)}{(2n+3)(2n+1)}\,P_{n+2}(\mu)+\frac{n\,(n+1)}{(2n+3)(2n-1)}\,P_n(\mu)+\frac{3}{2}\,\frac{n\,(n-1)}{(2n+1)(2n-1)}\,P_{n-2}(\mu),$$

being deduced from the formula*, $m \leqslant n$,

$$P_m(x)\,P_n(x)=\sum_{r=0}^{m}\frac{A_{m-r}\,.\,A_r\,.\,A_{n-r}}{A_{n+m-r}}\cdot\frac{2n+2m-4r+1}{2n+2m-2r+1}\,P_{n+m-2r}(x),$$

where $$A_m=\frac{1\,.\,3\,.\,5\,.\;\ldots\;\ldots\;(2m-1)}{m\,!}.$$

SECTION 5.

Multiply the above boundary condition, Section 4, by $P_n(\mu)$ and integrate between $+1$ and $-1$; then by the help of :

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0,\quad m \neq n,$$

and $$\int_{-1}^{1} [P_n(x)]^2\,dx = \frac{2}{2n+1},$$

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* Adams : *Proc. Royal Soc.*, Vol. XXVII.

we have

$$0 = R_n \cdot \Big[ n f_n(ka) \Big\{ 1 - 3\varepsilon \frac{n^3 - n^2 - 2n + 3}{(2n+3)(2n-1)} \Big\}$$

$$- (ka)^2 f_{n+1}(ka) \Big\{ 1 - 6\varepsilon \frac{n(n^2+n-1)}{(2n+3)(2n-1)} \Big\}$$

$$- 3\varepsilon \frac{n^2+n-1}{(2n+3)(2n-1)} (ka)^4 f_{n+2}(ka) \Big] (ka)^2$$

$+ S_n$ [a similar expression as above in which $f$ is replaced by $\psi$]

$$+ R_{n-2} \cdot \frac{3\varepsilon}{2} \cdot \frac{n(n-1)}{(2n-1)(2n-3)} \Big[ (n-2)^2 f_{n-2}(ka) - 2(n-2)(ka)^2 f_{n-1}(ka)$$

$$+ (ka)^4 f_n(ka) \Big]$$

$+ S_{n-2} \times$ [a similar expression as with $R_{n-2}$ in which $f$ is replaced by $\psi$]

$$+ R_{n+2} \cdot \frac{3\varepsilon}{2} (ka)^4 \frac{(n+1)(n+2)}{(2n+3)(2n+5)} [(n^2 - 6) f_{n+2}(ka)$$

$$- 2(n+2)(ka)^2 f_{n+3}(ka) + (ka)^4 f_{n+4}(ka)]$$

$+ S_{n+2} \times$ [a similar expression as with $R_{n+2}$ in which $f$ is replaced by $\psi$].

where to $n$ we may assign all values from 0 to $m$. ... (γ),

$$0 = R_{m-1} \cdot \frac{3\varepsilon}{2} \cdot \frac{(m+1)m}{(2m+1)(2m-1)} \times$$

$$[(m-1)^2 f_{m-1}(ka) - 2(m-1)(ka)^2 f_m(ka) + (ka)^4 f_{m+1}(ka)]$$

$+ S_{m-1} \times$ [a similar expression as with $R_{m-1}$ in which $f$ is replaced by $\psi$].

$$+ \varepsilon R'_{m+1} (ka)^2 [(m+1) f_{m+1}(ka) - (ka)^2 f_{m+2}(ka)]$$

$+ S'_{m+1} \times$ [a similar expression as with $R'_{m+1}$ in which $f$ is replaced by $\psi$] ... ... ... ... ... (δ)

and

$$0 = R_m \cdot \frac{3\varepsilon}{2} \cdot \frac{(m+1)(m+2)}{(2m+3)(2m+1)}$$

$$[m_2 f_m (ka) - 2m (ka)^2 f_{m+1} (ka) + (ka)^4 f_{m+2} (ka)]$$

$+ S_m \times$ [a similar expression as with $R_m$ in which $f$ is replaced by $\psi$]

$$+ R'_{m+1} \varepsilon . (ka)^3 [(m+2) f_{m+2} (ka) - (ka)^2 f_{m+3} (ka)]$$

$+ S'_{+m2} . \times$ [a similar expression as with $R'_{m+2}$ in which $f$ is replaced by $\psi$], ... ... $(\varepsilon)$

also

$$0 = R'_n [n f_n (ka) - (ka)^2 f_{n+1} (ka)]$$

$$+ S'_n [n \psi_n (ka) - (ka)^2 \psi_{n+1} (ka)], \quad \ldots \quad \ldots \quad (k)$$

where $n$ may have all values from $m + 3$ to infinity.

## SECTION 6.

From the $(m + 1)$ equations deduced from $(b)$ and the two equations deduced from the next by putting $n$ equal to $m+1$ and $m+2$, Section 3, combined with the $(m + 1)$ equations obtained from $(\gamma)$ and the two equations $(\delta)$ and $(\varepsilon)$, Section 5, the $(2m + 6)$ un-known co-efficients, namely, the $(m + 3)$ R's and $(m + 3)$ S's can be found by the method of solving linear equations. From the equations $(m)$ and $(k)$, it appears that for values of $n$ greater than $m+2$, the R's and S's are all zero. Thus the velocity potential of the motion within the contemplated space is given by

$$\phi = \sum_{n=0}^{m} P_n (\mu) (k\gamma)^n [R_n f_n (k\gamma) + S_n \psi_n (k\gamma)] e^{ikct}$$

$$+ \varepsilon \sum_{n=m+1}^{m+2} P_n (\mu) (k\gamma)^n [R'_n f_n (k\gamma) + S'_n \psi_n (k\gamma)] e^{ikct}$$

It should be noted that instead of taking the upper limit of $n$ in $\phi$, to be $m$, corresponding to the vibrations of the spherical surface, we may take any finite quantity $p$ ($> m$) to be the upper limit and thus the solution can be made to contain any number of terms we please, corresponding to the degree of approximation required.

# APPLICATION OF QUATERNION ANALYSIS TO FIND THE ASTATIC CENTRE OF CO-PLANAR FORCES.

By Bibhutibhusan Sen.

It is found that the quaternion analysis can be advantageously used in solving certain problems in mechanics of which the following is an example. The notation adopted is that of Gibbs so far as the vector and scalar products are concerned and the expression for the quaternion is taken from Kelland and Tait's *Introduction to Quaternions*: third edition, p. 37.

Let a body be acted on by co-planar forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \ldots \ldots \mathbf{F}_n$ with lines of action passing through the points whose position vectors relative to an origin O are $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \ldots \mathbf{r}_n$, respectively.

If we introduce forces $\pm \mathbf{F}_1, \pm \mathbf{F}_2, \pm \mathbf{F}_3 \ldots$ at O, then $+\mathbf{F}_1, +\mathbf{F}_2, +\mathbf{F}_3 \ldots +\mathbf{F}_n$ have a resultant

$$\mathbf{F} = \Sigma \mathbf{F}_1 \tag{1}$$

passing through O and the remaining forces form couples given by the vector products $\mathbf{r}_1 \times \mathbf{F}_1, \mathbf{r}_2 \times \mathbf{F}_2 \ldots \mathbf{r}_n \times \mathbf{F}_n$. These are equivalent to a single couple

$$\mathbf{G} = \Sigma\, \mathbf{r}_1 \times \mathbf{F}_1 \qquad \ldots \quad (2)$$

Now let all the forces be turned through their points of application through an angle $\phi$ in the positive direction. Then the force $\mathbf{F}_1$ becomes

$$\mathbf{F}'_1 = (\cos\phi + \varepsilon \sin\phi)\, \mathbf{F}_1$$

where $\varepsilon$ is a unit quadrantal versor having its line of application perpendicular to the plane of the forces.*

Hence if $\mathbf{F}'$ be the new resultant and $\mathbf{G}'$ the new couple,

$$\mathbf{F}' = \Sigma \mathbf{F}'_1 = \Sigma (\cos\phi + \varepsilon \sin\phi)\, \mathbf{F}_1$$

$$= (\cos\phi + \varepsilon \sin\phi)\, \Sigma \mathbf{F}_1 = (\cos\phi + \varepsilon \sin\phi)\, \mathbf{F} \qquad \ldots \quad (3)$$

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* Vide *Kelland and Tait*, p. 37.

and
$$\begin{aligned} \mathbf{G}' &= \Sigma\, \mathbf{r}_1 \times \mathbf{F}'_1 \\ &= \Sigma\, \mathbf{r}_1 \times (\cos\phi + \varepsilon \sin\phi)\, \mathbf{F}_1 \\ &= \Sigma\, (\mathbf{r}_1 \times \mathbf{F}_1) \cos\phi + \Sigma\, (\mathbf{r}_1 \times \varepsilon\, \mathbf{F}_1) \sin\phi \\ &= \mathbf{G} \cos\phi + \mathbf{V} \sin\phi \qquad \ldots \qquad \ldots \quad (4) \end{aligned}$$

where $\mathbf{V}$ stands for $\Sigma\, \mathbf{r}_1 \times \varepsilon\, \mathbf{F}_1$, and $\varepsilon\, \mathbf{F}_1$ is a force having the same magnitude on that of $\mathbf{F}_1$ but having its direction at right angles to it.

Hence $\mathbf{V}$ is the algebraic sum of the moments of forces when they have been turned through 90°. It is known as the *Virial* of the system.

Let $\mathbf{r}$ be the vector to a point O′ through which the resultant passes. Then the moment of the system about O′ must vanish.

$\therefore$
$$\Sigma\, (\mathbf{r}_1 - \mathbf{r}) \times \mathbf{F}'_1 = 0$$

*i.e.*,
$$\Sigma\, \mathbf{r}_1 \times \mathbf{F}'_1 - \Sigma\, \mathbf{r} \times \mathbf{F}'_1 = 0$$

*i.e.*,
$$\mathbf{G}' - \mathbf{r}\, \Sigma\, \mathbf{F}'_1 = 0.$$

Substituting from (3) and (4) this may be written :

$$\mathbf{G} \cos\phi + \mathbf{V} \sin\phi - \mathbf{r} \times (\cos\phi + \varepsilon \sin\phi)\, \mathbf{F} = 0$$

or
$$\cos\phi\, (\mathbf{G} - \mathbf{r} \times \mathbf{F}) + \sin\phi\, (\mathbf{V} - \mathbf{r} \times \varepsilon\, \mathbf{F}) = 0.$$

Here $\varepsilon\, \mathbf{F}$ is a vector $\perp$ to $\mathbf{F}$ and in the same plane. Let it be denoted by $\mathbf{H}$.

The above equation then becomes

$$\cos\phi\, (\mathbf{G} - \mathbf{r} \times \mathbf{F}) + \sin\phi\, (\mathbf{V} - \mathbf{r} \times \mathbf{H}) = 0 \quad \ldots \quad (5)$$

The equation (5) is a linear equation in $\mathbf{r}$ and represents a straight line. Whatever may be the value of $\phi$, it passes through a point given by the intersection of two lines

$$\mathbf{r} = \frac{\mathbf{G}}{\mathbf{F}} + t\mathbf{F} \quad \ldots \quad (6)$$

and
$$\mathbf{r} = \frac{\mathbf{V}}{\mathbf{H}} + t\mathbf{H}$$

The point of intersection P is the astatic centre of forces and is given by the vector

$$\overline{\mathrm{OP}} = \frac{\mathbf{G}}{\mathbf{F}} + \frac{\mathbf{V}}{\mathbf{H}}$$

It may be noted that if the forces are to be in equilibrium in the displaced position also, then $\mathbf{F}'$ and $\mathbf{G}'$ must both vanish. From (3) and (4) we see that these conditions require that

$$\mathbf{F} = 0,\ \mathbf{G} = 0 \text{ and } \mathbf{V} = 0$$

These are the conditions of astatic equilibrium

# ON A CERTAIN LIMIT CONNECTED WITH PAIRS OF INTEGERS

BY S. D. CHOWLA,
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**1.** Let us denote by $\{m, n\}$ the number of distinct pairs of integers, $r$ and $s$, such that*

$$(r, s) = 1,$$
$$1 \leqslant r \leqslant m,$$
$$1 \leqslant s \leqslant n.$$

Prof. Narasinga Rao has conjectured† that the ratio

$$\frac{\{m, n\}}{mn}$$

approaches $\frac{6}{\pi^2}$ as $m$ and $n$ tend simultaneously to infinity *in any manner whatsoever.* I give below a proof of this result.

**2.** Denote by $\phi(r, n)$ the number of positive integers less than $n$ and prime to $r$; by $\phi(n)$ the number of positive integers less than and prime to $n$; by $\varpi(x)$ the number of distinct prime factors of $x$; by $\mathrm{E}(x)$ the greatest integer contained in $x$; by $\mu(x)$ the Mobiüs function which is zero if $x$ has any repeated factors, and is equal to $+1$ or $-1$ according as $x$ has an even or an odd number of dissimilar prime factors; by $\mathrm{K}_r$ $(r = 1, 2, 3, \ldots)$ a real number whose absolute value does not exceed unity.

If $a, b, c, \ldots$ are the different prime factors of $r$, it is obvious that

$$\phi(r, n) = \mathrm{E}(n) - \Sigma\,\mathrm{E}\left(\frac{n}{a}\right) + \Sigma\,\mathrm{E}\left(\frac{n}{ab}\right) - \Sigma\,\mathrm{E}\left(\frac{n}{abc}\right) + - \ldots$$

whence
$$\phi(r, n) = n \cdot \frac{\phi(r)}{r} + \mathrm{K}_1 \cdot 2^{\varpi(r)} \qquad \ldots \quad (1)$$

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* The symbol $(x, y)$ denotes the greatest common divisor of $x$ and $y$.

† *Jour. Ind. Math. Soc.*, Vol 17, No. 7, 100–102.

From our definitions we get

$$\{m, n\} = \phi(1, n) + \phi(2, n) + \dots\dots + \phi(m, n)$$

$$= \phi(1, m) + \phi(2, m) + \dots\dots + \phi(n, m) \qquad \dots \quad (2)$$

From (1) and (2) we get

$$\{m, n\} = n.\left\{\sum_{x=1}^{m} \frac{\phi(x)}{x}\right\} + K_2\left\{\sum_{1}^{m} 2^{\varpi(x)}\right\} \qquad \dots \quad (3)$$

$$= m.\left\{\sum_{x=1}^{n} \frac{\phi(x)}{x}\right\} + K_3\left\{\sum_{1}^{n} 2^{\varpi(x)}\right\}$$

Now*

$$\frac{\phi(x)}{x} = \sum_{d/x} \frac{\mu(d)}{d}$$

Hence

$$\sum_{x=1}^{n} \frac{\phi(x)}{x} = \sum_{1}^{n} \frac{\mu(x)}{x} E\left(\frac{n}{x}\right) \qquad \dots \quad (4)$$

But

$$E\left(\frac{n}{x}\right) = \frac{n}{x} + K_4$$

hence (4) gives

$$\sum_{1}^{n} \frac{\phi(x)}{x} = n.\left\{\sum_{1}^{n} \frac{\mu(x)}{x^2}\right\} + K_5\left\{\sum_{1}^{n} \frac{1}{x}\right\}$$

$$= n\left\{\sum_{1}^{\infty} \frac{\mu(x)}{x^2}\right\} + K_5(1 + \log n) + K_6\left\{\sum_{n+1}^{\infty} \frac{1}{x^2}\right\}$$

$$= \frac{6n}{\pi^2} + K_5(\log n) + 2K_7 \qquad \dots \quad \dots \quad (5)$$

* The symbol $d/n$ means that $d$ is a divisor of $n$.

Similarly

$$\sum_{1}^{n} 2^{\varpi(x)} = 3K_8\, n \log (2en) \quad \ldots \quad \ldots \quad (6)$$

From (3), (5) and (6) we get

$$\{m. n\} = \frac{6mn}{\pi^2} + K_9\,(m \log n) + 2K_{10}\, m + 3K_{11}\, n \log (2en)$$

$$= \frac{6mn}{\pi^2} + K_{12}\,(n \log m) + 2K_{13}\, n + 3K_{14}\, m \log (2em).$$

Hence

$$\frac{\{m, n\}}{mn} = \frac{6}{\pi^2} + K_9 \frac{\log n}{n} + \frac{2K_{10}}{n} + 3K_{11} \frac{\log (2en)}{m}$$

$$= \frac{6}{\pi^2} + K_{12} \frac{\log m}{m} + \frac{2K_{13}}{m} + 3K_{14} \frac{\log (2em)}{n} \ldots \quad (7)$$

Now at least one of the quantities

$$3K_{11} \frac{\log (2en)}{m} \text{ and } 3K_{14}. \frac{\log (2em)}{n}$$

approaches zero as $m$ and $n$ tend simultaneously to $\infty$, because their product does not exceed

$$9. \frac{\log (6m)}{m}. \frac{\log (6n)}{n}$$

in absolute value; that is, their product tends to 0 as $m$ and $n$ tend to $\infty$. Hence in (7) at least one of the two expressions for $\frac{\{m, n\}}{mn}$ obviously approaches $\frac{6}{\pi^2}$ as $m$ and $n$ *tend simultaneously to infinity in an arbitrary manner*.

# ON THE PERIODIC SOLUTIONS OF THE DIFFERENTIAL EQUATION $\frac{d^2y}{dx^2} + (A + 16q \cos 2nx)\, y = 0$.

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## 1. Introduction

The object of this paper is to investigate the different orders of the periodic solutions of the differential equation

$$\frac{d^2y}{dx^2} + (A + 16q \cos 2nx)\, y = 0, \quad \ldots \qquad \ldots \quad (1)$$

where $n$ is any integer, $A = m^2 + a'$, $m$ being any integer* including zero, $a'$ an arbitrary constant, and $q$ a small quantity so that its higher powers may be neglected.

It will be seen that some of these solutions correspond to Mathieu Functions arising out of the equation,

$$\frac{d^2y}{dz^2} + (A' + 16q' \cos 2z)\, y = 0 \ldots \qquad \ldots \quad (2)$$

For $n = 1$, the differential equation (1) is nothing but Mathieu's equation; otherwise it differs† from the latter. Thus, let us change the independent variable $z$ to $x$ by the relation $z = nx$; the equation (2), then, assumes the form

$$\frac{d^2y}{dx^2} + n^2 (A' + 16q' \cos 2nx)\, y = 0.$$

Now $n^2q'$ may be large even if $q'$ be small and hence its higher powers cannot be neglected.

It may be remarked that the differential equation is a particular case of Hill's equation and the possibility of the solutions of different orders

* $m$ denotes the order of the solution.

† In the sense that we have to take to asymptotic solution. This is done with a view to investigate certain properties shown in this and the other papers of this series.

has already been given by Lindsay Ince* ; but there are some interesting points about the solutions of (i) which to the knowledge of the writer have not yet been investigated.

## 2. † Method of solving the equation.

We shall deduce the different periodic solutions, both odd and even, with the help of integral equations used by Dr. S. C. Dhar ‡ in finding the solutions of (2).

Let us denote the even and odd periodic solutions of the differential equation by

$$c_ne_1(x, q),\ c_ne_2(x, q) \ldots \quad \ldots \quad \ldots\ c_ne_m(x, q)$$

and $$s_ne_1(x, q),\ s_ne_2(x, q) \ldots \quad \ldots \quad \ldots\ s_ne_m(x, q)$$

respectively, in analogy with Mathieu Functions, so that $c_ne_m(x, q)$ and $s_ne_m(x, q)$ will respectively denote the even and odd periodic solutions of order $m$.

It has been shown by S. C. Dhar ‡ that if Q and R be given functions of $z$ and $y_1(z)$ and $y_2(z)$ be the even and odd solutions of

$$\frac{d^2y}{dz^2} + Qy = 0,$$

then the even and odd solutions of

$$\frac{d^2y}{dz^2} + Qy = Ry,$$

will be given by the integral equations

$$y(z) = By_1(z) + \frac{1}{C}\int_0^z R(t) \begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z) \end{vmatrix} y(t)\, dt,$$

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* E. Lindsay Ince : "On a General Solution of Hill's equation." ***Monthly Notices of the Royal Astronomical Society***, Vol. LXXV, Pages **436—448.** In this paper the author completely solves the differential equation : $d^2y / dz^2 + (\theta_0 + 2\theta_1 \cos 2z + 2\theta_2 \cos 4z + \ldots \ldots)\, y = 0$, where $\theta_0 = 1 +$ a function of $\theta_1, \theta_2, \theta_3$ ...... etc.; and concludes the paper thus : 'It is however, equally possible to obtain similar expansions which are suited to the cases in which the quantity $\theta_0$ approches the values 4, 9, 16 ... ... etc,"

† So far the writer is aware, this method, first given by Priestley (***Proc. Lond. Math. Soc.*** Vol XX) has only been applied to Mathieu's Differential Equation $d^2y / dz^2 + (A + 16q \cos 2z)\, y = 0$. See, S. C. Dhar, "On certain integral equations of the second kind and the construction of Mathieu Functions, ***Bulletin of the Cal. Math. Soc.***, Vol, XVIII (1927),

‡ *Loc. cit,*

and $$y(z) = \mathrm{B} y_2(z) + \frac{1}{\mathrm{C}} \int_0^z \mathrm{R}(t) \begin{vmatrix} y_1(t) & y_2(t) \\ y_1(z) & y_2(z) \end{vmatrix} y(t)\, dt,$$

respectively, where B is an arbitrary constant and

$$\mathrm{C} = y_1 \frac{dy_2}{dz} - y_2 \frac{dy_1}{dz}.$$

Putting now $\mathrm{A} = m^2 + a'$, we write the equation (1) in the form

$$\frac{d^2y}{dx^2} + m^2 y = -(a' + 16q \cos 2nx)\, y.$$

Applying the above method, we see that the even and odd solutions of (i) are respectively given by

$$\psi(x) = \mathrm{B} \cos mx + \frac{1}{m} \int_0^x (a' + 16\, q \cos 2nt) \sin m(t-x)\, \psi(t)\, dt,$$

and

$$\psi(x) = \mathrm{B} \sin mx + \frac{1}{m} \int_0^x (a' + 16\, q \cos 2nt) \sin m(t-x)\, \psi(t)\, dt.$$

Suppose

$$\psi(x) = \psi_0(x) + q\, \psi_1(x) + q^2\, \psi_2(x) + \dots \dots,$$

where $\psi_0, \psi_1, \psi_2 \dots \dots$ are periodic functions of $x$ and do not contain $q$, and suppose also

$$a' = a_1 q + a_2 q^2 + \dots \dots$$

and $$\mathrm{B} = 1 + b_1 q + b_2 q^2 + \dots \dots$$

where $a_1, a_2 \dots b_1, b_2 \dots$ are constants.

Substituting these values we find that the even and odd solutions are given by

$$\psi_0(x) + q\, \psi_1(x) + q^2\, \psi_2(x) + \dots = (1 + b_1 q + b_2 q^2 + \dots) \cos mx$$
$$+ \frac{1}{m} \int_0^x [\{(a_1 q + a_2 q^2 + \dots) + 16q \cos 2nt\} \times$$
$$\sin m(t-x) \{\psi_0(t) + q\, \psi_1(t) + q^2\, \psi_2(t) + \dots \dots\}]\, dt$$

and

$$\psi_0(x) + q\, \psi_1(x) + q^2\, \psi_2(x) + \dots = (1 + b_1 q + b_2 q^2 + \dots) \sin mx +$$
$$\frac{1}{m} \int_0^x [\{(a_1 q + a_2 q^2 + \dots \dots) + 16q \cos 2nt\} \times$$
$$\sin m(t-x)\ \{\psi_0(t) + q\, \psi_1(t) + q^2\, \psi_2(t) + \dots \dots\}]\, dt$$

respectively.

Hence equating the co-efficients of successive powers of $q$, we get for the even solution

$$\psi_0(x) = \cos mx$$

$$\psi_1(x) = b_1 \cos mx + \frac{1}{m}\int_0^x (a_1 + 16 \cos 2nt) \sin m(t-x)\, \psi_0(t)\, dt$$

$$\psi_2(x) = b_2 \cos mx + \frac{a_2}{m}\int_0^x \sin m(t-x)\, \psi_0(t)\, dt$$

$$+ \frac{1}{m}\int_0^x (a_1 + 16 \cos 2nb) \sin m(t-x)\, \psi_1(t)\, dt,$$

... ... ... ... ... ...

and for an odd solution

$$\psi_0(x) = \sin mx$$

$$\psi_1(x) = b_1 \sin mx + \frac{1}{m}\int_0^x (a_1 + 16 \cos 2nt) \sin m(t-x)\, \psi_0(t)\, dt,$$

$$\psi_2(x) = b_2 \sin mx + \frac{a_2}{m}\int_0^x \sin m(t-x)\, \psi_0(t)\, dt +$$

$$\frac{1}{m}\int_0^x (a_1 + 16 \cos 2nt) \sin m(t-x)\, \psi_1(t)\, dt.$$

... ... ... ... ... ...

so that we have only to put $m = 1, 2, 3 \ldots \ldots$ to get the periodic solutions of different orders.

The constants $a_1, a_2 \ldots \ldots$ will be determined from the fact that $\psi_0, \psi_1, \psi_2 \ldots \ldots$ must be periodic. The values of $b_1, b_2 \ldots$ etc. will be determined from the condition that $\psi_1, \psi_2 \ldots \ldots$ will not contain $\cos mx$ or $\sin mx$, according as the solution under investigation, is even, or odd.

## 3. Even Solutions of the Differential Equation.

Putting $m = 1$, we shall get the form for $e_n c_1(x, q)$.

In this case,

$$\psi_0(x) = \cos x$$

$$\psi_1(x) = b_1 \cos x + \int_0^x (a_1 + 16 \cos 2nt) \sin (t-x) \cos t\, dt$$

$$= \left[b_1 - \frac{4}{n^2-1}\right] \cos x - \frac{a_1}{2}\, x \sin x + \frac{2 \cos (2n+1)\, x}{n\,(n+1)}$$

$$+ \frac{2 \cos (2n-1)\, x}{n\,(n-1)} .$$

$\therefore$ If $b_1 = 4/(n^2-1)$, and $a_1 = 0$, then

$$\psi_1(x) = \frac{2\cos(2n+1)x}{n(n+1)} + \frac{2\cos(2n-1)x}{n(n-1)}.$$

$$\psi_2(x) = b_2\cos x + a_2\int_0^x \sin(t-x)\cos t\,dt +$$

$$\int_0^x \left[(a_1 + 16\cos 2nt)\sin(t-x)\left\{\frac{2\cos(2n+1)t}{n(n+1)} + \frac{2\cos(2n-1)t}{n(n-1)}\right\}\right]dt.$$

$$= \left[b_2 - \frac{4(2n^2+1)}{n^2(n^2-1)(4n^2-1)}\right]\cos x + \left[-\frac{a_1}{2} - \frac{16}{n^2-1}\right]$$

$$x\sin x + \frac{2\cos(4n+1)x}{n^2(n+1)(2n+1)} + \frac{2\cos(4n-1)x}{n^2(n-1)(2n-1)}.$$

If $b_2 = \frac{4(2n^2+1)}{n^2(n^2-1)(4n^2-1)}$, $a_2 = -\frac{32}{n^2-1}$, then

$$\psi_2(x) = \frac{2\cos(4n+1)x}{n^2(n+1)(2n+1)} + \frac{2\cos(4n-1)x}{n^2(n-1)(2n-1)}.$$

*Hence a solution of the differential equation (1) is given by*

$$\psi(x) = c_n e_1(x, q) = \cos x + q\left\{\frac{2\cos 2n+1\,x}{n(n+1)} + \frac{2\cos(2n-1)x}{n(n-1)}\right\}$$

$$+ q^2\left\{\frac{2\cos(4n+1)x}{n^2(n+1)(n+2)} + \frac{2\cos(4n-1)x}{n^2(n-1)(2n-1)}\right\} + - \ldots \quad (3.1)$$

... ... ... ... ... ...

*subject to the condition that*

$$A = 1 - \frac{32}{n^2-1}q^2 \;\ldots\;\ldots$$

Let us next find $c_n e_2(x, q)$. For this, we have

$$\psi_0(x) = \cos 2x$$

$$\psi_1(x) = b_1\cos 2x + \tfrac{1}{2}\int_0^x (a_1 + 16\cos 2nt)\sin 2(t-x)\cos 2t\,dt$$

$$= \left[b_1 - \frac{4}{n^2-4}\right]\cos 2x - \frac{a_1}{4}x\sin 2x$$

$$+ \frac{2 \cos (2n + 2) x}{n (n + 2)} + \frac{2 \cos (2n - 2) x}{n (n - 2)}.$$

Taking $a_1 = 0$, $b_1 = \frac{4}{n^2 - 4}$, we get

$$\psi_1 (x) = \frac{2 \cos (2n + 2) x}{n (n + 2)} + \frac{2 \cos (2n - 2) x}{n (n - 2)}$$

$$\psi_2 (x) = b_2 \cos 2x + \frac{a_2}{2} \int_0^x \sin 2 (t - x) \cos 2t \, dt$$

$$+ \tfrac{1}{2} \int_0^x \Big[ (a_1 + 16 \cos 2nt) \sin 2 (t - x)$$

$$\left\{ \frac{2 \cos (2n + 2) t}{n (n + 2)} + \frac{2 \cos (2n - 2) t}{n (n - 2)} \right\} \Big] dt,$$

$$= \left[ b_2 - \frac{4 (2n^2 + 4)}{n^2 (n^2 - 4) (4n^2 - 4)} \right] \cos 2x + \left[ -\frac{a_2}{4} - \frac{8}{n^2 - 4} \right] x \sin 2x$$

$$+ \frac{2 \cos (4n + 2) x}{n^2 (n + 2) (2n + 2)} + \frac{2 \cos (4n - 2)}{n^2 (n - 2) (2n - 2)}.$$

If $b_2 = \frac{4 (2n^2 + 4)}{n^2 (n^2 - 4) (4n^2 - 4)}$, $a_2 = -\frac{32}{n^2 - 4}$,

$$\psi_2 (x) = \frac{2 \cos (4n + 2) x}{n^2 (n + 2) (2n + 2)} + \frac{2 \cos (4n - 2) x}{n^2 (n - 2) (2n - 2)}$$

Hence

$$\psi (x) = c_n e_2 (x, q) = \cos 2x + q \left\{ \frac{2 \cos (2n + 2) x}{n (n + 2)} \right.$$

$$\left. + \frac{2 \cos (2n - 2) x}{n (n - 2)} \right\} + q^2 \left\{ \frac{2 \cos (4n + 2) x}{n^2 (n + 2) (2n + 2)} \right.$$

$$\left. + \frac{2 \cos (4n - 2) x}{n^2 (n - 2) (2n - 2)} \right\} + \quad \ldots \qquad \ldots \quad (3.2)$$

with

$$A = 4 - \frac{32}{n^2 - 4} q^2 \ldots \ldots \ldots$$

The expressions for $c_n e_1 (x, q)$ and $c_n e_2 (x, q)$ are quite analogous and suggest the forms of the solutions of higher orders. Thus we may

show that

$$c_n e_3(x, q) = \cos 3x + q\left\{\frac{2\cos(2n+3)x}{n(n+3)} + \frac{2\cos(2n-3)x}{n(n-3)}\right\}$$

$$+ q^2\left\{\frac{2\cos(4n+3)x}{n^2(n+3)(2n+3)} + \frac{2\cos(4n-3)x}{n^2(n-3)(2n-3)}\right\} + \ldots \quad (3.3)$$

A being equal to

$$9 - \frac{32}{n^2 - 9} q^2 \ldots\ldots$$

On working out, it is found that $c_n e_r(x, q)$, the even periodic solution of $r$th order, is given by

$$c_n e_r(x, q) = \cos rx + q\left\{\frac{2\cos(2n+r)x}{n(n+r)} + \frac{2\cos(2n-r)x}{n(n-r)}\right\}$$

$$+ q^2\left\{\frac{2\cos(4n+r)x}{n^2(n+r)(2n+r)} + \frac{2\cos(4n-r)x}{n^2(n-r)(2n-r)}\right\} + \ldots$$

where

$$A = r^2 - \frac{32}{n^2 - r^2} q^2 \ldots\ldots\ldots$$

An interesting solution is afforded by the function $c_n e_n(x, q)$ which now, we shall deduce.

For this, evidently

$$\psi_0(x) = \cos nx$$

$$\psi_1(x) = b_1 \cos nx + \frac{1}{n}\int_0^x (a_1 + 16\cos 2nt)\sin n(t-x)\cos nt\, dt.$$

$$= \left[b_1 - \frac{1}{n^2}\right]\cos nx + \left[-\frac{a_1}{2n} - \frac{4}{n}\right] x \sin nx + \frac{\cos 3nx}{n^2}$$

$\therefore$ Putting $a_1 = -8$, $b_1 = \frac{1}{n^2}$,

$$\psi_1(x) = \frac{\cos 3nx}{n^2}$$

$$\psi_2(x) = b_2 \cos 2nx + \frac{a_2}{n}\int_0^x \sin n(t-x)\cos nt\, dt$$

$$+ \frac{1}{n}\int_0^x (a_1 + 16\cos 2nt)\sin n(t-x)\frac{\cos 3nt}{n^2}\, dt$$

$$= \left[ b_2 + \frac{2}{3n^4} \right] \cos nx + \left[ -\frac{a_2}{2n} - \frac{4}{n^3} \right] x \sin nx$$

$$- \frac{1}{n^4} \cos 3nx + \frac{1}{3n^4} \cos 5nx$$

Putting $\quad b_2 = -\frac{2}{3n^4}, \; a_2 = -\frac{8}{n^2},$

$$\psi_2(x) = -\frac{1}{n^4} \cos 3nx + \frac{1}{3n^4} \cos 5nx.$$

Hence

$$\psi(x) = c_n e_n(x, q) = \cos nx + q \cdot \frac{\cos 3nx}{n^2} q^2 + \left\{ -\frac{\cos 3nx}{n^4} + \frac{\cos 5nx}{3n^4} \right\} + \ldots$$

with

$$A = n^2 - 8q - \frac{8}{n^2} q^2 \ldots \ldots$$

Here it may be noticed that for $n = 1$, $c_n e_n(x, q)$ is nothing but $ce_1(x, q)$, the even Mathieu Function of order 1. Moreover, we see that we can get to a good approximation to the function by taking the first few terms. Thus, if

$$q = \frac{1}{10}, \; n = 10, \; \frac{q^2}{N^4} = \frac{1}{1000000}$$

Another interesting solution is given by $c_n e_{2n}(x, q)$.

Here

$$\psi_0(x) = \cos 2nx$$

$$\psi_1(x) = b_1 \cos 2nx + \frac{1}{2n} \int_0^x (a_1 + 16 \cos 2nt) \sin 2n(t - x) \cos 2nt \, dt$$

$$= \left[ b_1 + \frac{4}{3n^2} \right] \cos 2nx - \frac{a_1}{4n} x \sin 2nx + \frac{2 \cos 4nx}{3n^2} - \frac{2}{n^2}$$

$\therefore \quad$ If $b_1 = -\frac{4}{3n^2}, \; a_1 = 0,$

$$\psi_1(x) = \frac{2}{3n^2} \cos 4nx - \frac{2}{n^2}$$

$$\psi_2(x) = b_2 \cos 2nx + \frac{a_2}{2n} \int_0^x \sin 2n(t - x) \cos 2nt \, dt$$

$$+\frac{1}{2n}\int_0^x \left[(a_1+16\cos 2nt)\sin 2n(t-x)\left\{\frac{2}{3n^2}\cos 4nt-\frac{2}{n^2}\right\}\right]dt$$

$$=\left[b_2-\frac{1}{6n^4}\right]\cos 2nx+\left[\frac{a_2}{4n}-\frac{20}{3n^3}\right]x\sin 2nx+\frac{\cos 6nx}{6n^4}$$

$$\therefore \quad \text{If } b_2=\frac{1}{6n^4},\ a_2=\frac{80}{3n^2},$$

$$\psi_2(x)=\frac{\cos 6nx}{6n^4}.$$

Hence we have

$$\psi(x)=c_ne_{2n}(x,q)=\cos 2nx+q\left\{\frac{2}{3n^2}\cos 4nx-\frac{2}{n^2}\right\}+q^2\cdot\frac{\cos 6nx}{6n^4}$$

$$+\ \dots\ \dots \qquad (3.5)$$

with $$A=4n^2+\frac{80}{3n^2}q^2+\ \dots\ \dots$$

The forms of $c_ne_n(x,q)$ and $c_ne_{2n}(x,q)$ show that, for $n=1$, they are equal to $ce_1(x,q)$ and $ce_2(x,q)$, the even Mathieu Function of orders 1 and 2 respectively. The analogy traced in (3.4) and 3.5) shows that, for $n=1$, $c_ne_{3n}(x,q)$, $c_ne_{4n}(x,q)$, ... ..., will be nothing but even Mathieu Functions of order 3, 4 ... ... . Moreover it is easily seen that the analogy holds up to the general term so that if we put $n=1$ in $c_ne_{rn}(x,q)$, we shall get $ce_r(x,q)$, the even Mathieu Function of order $r$. But between $c_ne_{rn}(x,q)$ and $c_ne_{(r+1)n}(x,q)$ there are $(n-1)$ solutions of the differential equation (1) to which correspond no Mathieu Functions.

## 4. Odd Solutions of the Differential Equation.

We shall find first the odd solution of order 1, denoted by $s_ne_1(x,q)$.

As before,

$$\psi_0(x)=\sin x$$

$$\psi_1(x)=b_1\sin x+\int_0^x(a_1+16\cos 2nt)\sin(t-x)\sin t\,dt$$

$$=\left[b_1-\frac{a_1}{2}+\frac{4}{n^2-1}\right]\sin x+\frac{a_1}{2}x\cos x+\frac{2\sin(2n+1)x}{n(n+1)}$$

$$-\frac{2\sin(2n-1)x}{n(n-1)}.$$

$\therefore$ If $a_1 = 0,\ b_1 = -\dfrac{4}{n^2-1}$,

$$\psi_1(x) = \frac{2\sin(2n+1)x}{n(n+1)} - \frac{2\sin(2n-1)x}{n(n-1)}.$$

$$\psi_2'(x) = b_2 \sin x + a_2 \int_0^x \sin(t-x)\sin t\,dt$$

$$+ \int_0^x \left[(a_2 + 16\cos 2nt)\sin(t-x)\left\{\frac{2\sin(2n+1)t}{n(n+1)} - \frac{2\sin(2n-1)t}{n(n-1)}\right\}\right] dt.$$

$$= \left[b_2 - \frac{a_2}{2} + \frac{4(10n^2-1)}{n^2(n^2-1)(4n^2-1)}\right]\sin x + \left[\frac{a_2}{2} + \frac{16}{n^2-1}\right] x\cos x$$

$$+ \left\{\frac{2\sin(4n+1)x}{n^2(n+1)(2n+1)} - \frac{2\sin(4n-1)x}{n^2(n-1)(2n-1)}\right\}$$

$\therefore$ If $a_2 = -\dfrac{32}{n^2-1},\quad b_2 = -\dfrac{4(10n^2-1)}{n^2(n^2-1)(4n^2-1)}$,

$$\psi_2(x) = \frac{2\sin(4n+1)x}{n^2(n+1)(2n+1)} - \frac{2\sin(4n-1)x}{n^2(n-1)(2n-1)}$$

$$\therefore\quad \psi(x) = s_n e_1(x, q)$$

$$= \sin x + q\left\{\frac{2\sin(2n+1)x}{n(n+1)} - \frac{2\sin(2n-1)x}{n(n-1)}\right\}$$

$$+ q^2\left\{\frac{2\sin(4n+1)x}{n^2(n+1)(2n+1)} - \frac{2\sin(4n-1)x}{n^2(n-1)(2n-1)}\right\} + \dots \qquad \dots (4.1)$$

where

$$A = 1 - \frac{32}{n^2-1}q^2 \dots\dots\dots$$

Proceeding as above, we can show that

$$s_n e_2(x, q) = \sin 2x + q\left\{\frac{2\sin(2n+2)x}{n(n+2)} - \frac{2\sin(2n-2)x}{n(n-2)}\right\}$$

$$+ q^2\left\{\frac{2\sin(4n+2)x}{n^2(n+2)(2n+2)} - \frac{2\sin(4n-2)x}{n^2(n-2)(2n-2)}\right\} + \dots$$



195331

with

$$A \doteq 4 - \frac{32}{n^2 - 4} q^2 \ldots$$

It is quite evident that the expressions for the solutions of higher orders, can, at once, be written down from analogy.

An interesting odd solution is given by $A_n e_n (x, q)$. Here

$$\psi_0(x) = \sin nx$$

$$\psi_1(x) = b_1 \sin nx + \frac{1}{n} \int_0^x (a_1 + 16 \cos 2nt) \sin n(t - x) \sin nt \, dt$$

$$= \left[b_1 - \frac{3}{n^2}\right] \sin nx + \left[\frac{a_1}{2n} - \frac{4}{n}\right] x \cos nx + \frac{\sin 3nx}{n^2}.$$

$\therefore$ If $a_1 = 8$, $b_1 = \frac{3}{n^2}$,

$$\psi_1(x) = \frac{\sin 3nx}{n^2}$$

$$\psi_2(x) = b_2 \sin nx + \frac{a_2}{n} \int_0^x \sin n(t - x) \sin nt \, dt$$

$$+ \frac{1}{n} \int_0^x (a_1 + 16 \cos 2nt) \sin n(t - x) \frac{\sin 3nt}{n^2} dt$$

$$= \left[b_2 - \frac{11}{3n^4}\right] \sin nx + \left[\frac{a_2}{2n} + \frac{4}{n^3}\right] x \cos nx$$

$$+ \frac{\sin 3nx}{n^4} + \frac{\sin 5nx}{3n^4}$$

$\therefore$ If $a_2 = -\frac{8}{n^2}$, $b_2 = \frac{11}{3n^4}$,

$$\psi_2(x) = \frac{\sin 3nx}{n^4} + \frac{\sin 5nx}{3n^4}.$$

Hence

$$\psi(x) = s_n e_n (x, q) = \sin nx + q \cdot \frac{\sin 3nx}{n^2} + q^2 \left(\frac{\sin 5nx}{3n^4} + \frac{\sin 3nx}{n^4}\right) + \ldots \quad \ldots \quad (4.2)$$

subject to the condition that

$$A = n^2 + 8q - \frac{8}{n^2} \cdot q^2 \ldots \ldots$$

It is clear that, for $n = 1$, $_se_n\,(x, q)$ is nothing but $se_1\,(x, q)$, the odd Mathieu Function of order 1. This analogy is carried on to the solutions of higher orders, as in the case of even solutions. Thus $_ne_{2n}\,(x, q)$, $s_ne_{3n}\,(x, q)$ ... ... ... etc, for $n = 1$, exactly equal to $se_2\,(x, q)$, $se_3\,(x, q)$ ... etc. Hence for odd solutions as well there are $(n-1)$ solutions lying between those that correspond to odd Mathieu Functions.

We shall now give $s_n\,e_r\,(x, q)$, the odd solution of order $r$.

For this,

$$\psi_0\,(x) = \sin rx$$

$$\psi_1\,(x) = b_1 \sin rx + \frac{1}{r}\int_0^x (a_1 + 16 \cos 2nt) \sin r\,(t - x) \sin rt\, dt.$$

$$= \left[b_1 - \frac{a_1}{2r^2} + \frac{4}{n^2 - r^2}\right] \sin rx + \frac{a_1}{2r}\, x \cos rx +$$

$$\frac{2 \sin (2n + r)\, x}{n\,(n + r)} - \frac{2 \sin (2n - r)\, x}{n\,(n - r)}\,.$$

$$\therefore \quad \text{If } a_1 = 0, \quad b_1 = -\frac{4}{n^2 - r^2}\,,$$

$$\psi_1 (x) = \frac{2 \sin (2n + r)\, x}{n\,(n + r)} - \frac{2 \sin (2n - r)\, x}{n\,(n - r)}\,,$$

$$\psi_2\,(x) = b_2 \sin rx + \frac{a_2}{r}\int_0^x \sin r\,(t - x) \sin rt\, dt +$$

$$\frac{1}{r}\int_0^x (a_1 + 16 \cos 2nt) \sin r\,(t - x) \times$$

$$\left\{\frac{2 \sin (2n + r)\, t}{n\,(n + r)} - \frac{2 \sin (2n - r)\, t}{n\,(n - r)}\right\} dt$$

$$= \left[\frac{a_2}{2r} + \frac{16}{r\,(n^2 - r^2)}\right] x \cos rx +$$

$$\left[b_2 - \frac{a_2}{2r^2} - \frac{16}{r^3\,(n^2 - r^2)} + \frac{4\,(10n^2 - r^2)}{n^2\,(n^2 - r^2)\,(4n^2 - r^2)}\right] \sin rx$$

$$+ \frac{2 \sin (4n + r)\, x}{n^3\,(n + r)\,(2n + r)} - \frac{2 \sin (4n - r)\, x}{n^2\,(n - r)\,(2n - r)}\,.$$

$$\therefore \text{ If } a_2 = -\frac{32}{n^2 - r^2}, \quad b_2 = -\frac{4(10n^2 - r^2)}{n^2(n^2 - r^2)(4n^2 - r^2)},$$

$$\psi_2(x) = \frac{2\sin(4n + r)x}{n^2(n + r)(2n + r)} - \frac{2\sin(4n - r)x}{n^2(n - r)(2n - r)}.$$

Hence

$$\psi(x) = {}_{s_n}e_r(x, q) = \sin rx + q\left\{\frac{2\sin(2n + r)x}{n(n + r)} - \frac{2\sin(2n - r)x}{n(n - r)}\right\}$$

$$+ q^2\left\{\frac{2\sin(4n + r)x}{n^2(n + r)(2n + r)} - \frac{2\sin(4n - r)x}{n^2(n - r)(2n - r)}\right\} + \ldots\ldots, \quad \ldots (4.3)$$

where

$$A = r^2 - \frac{32}{n^2 - r^2}q^2 \ldots\ldots$$

It is easily seen that, for $n = 1$,

$$s_ne_r(x, q) = se_r(x, q)$$

It is also evident that the various odd functions are rapidly convergent series for larger values of $n$ and smaller values of $q$.

## 5. Expression for $c_ne_0(x, q)$.

Having established the three fundamental properties of the periodic solutions of the differential equation, *viz.*,

(1) that the solutions are quite analogous and suggest the forms of the solutions of higher orders but of the same kind,

(2) that there are $(n - 1)$ solutions of the differential equation which lie between the solutions that correspond to Mathieu Functions,

(3) that the solutions are rapidly convergent series,

we proceed to utilize the last two in finding the expression for $c_ne_0(x, q)$, the solution of order zero, which, hence, correspond, to $ce_0(x, q)$, Mathieu Function of order zero. The comparison of the different values of A for which $c_ne_n(x, q)$ and $c_ne_{2n}(x, q)$ are given and of the different corresponding values of A for $ce_1(x, q)$ and $ce_2(x, q)$ suggests that the value of A for $c_ne_0(x, q)$ in analcgy with the value of A in $ce_0(x, q)$ will

be of the form

$$A = -32\frac{q^2}{n^2} + 224\frac{q^4}{n^6} - \frac{2^{10}.29}{9}\frac{q^6}{n^{10}} + \dots \dots$$

Let us take the solution $c_n e_0(x, q)$ of the differential equation (1) in the form

$$y = 1 + q f_1(x) + q^2 f_2(x) + q^3 f_3(x) + \dots$$

where $f_1(x), f_2(x)$, ... etc. are functions of $x$ only and do not contain $q$.

Substituting the value of $y$ and A in (1), we have

$$q f_1''(x) + q^2 f_2''(x) + q^3 f_3''(x) + \dots \dots$$

$$+ \left\{ -32\frac{q^2}{n^2} + 224\frac{q^4}{n^6} - \frac{2^{10}.29}{9}\frac{q^6}{n^{10}} \dots \dots + 16\, q \cos 2nx \right\} \times$$

$$\{ 1 + q f_1(x) + q^2 f_2(x) + q^3 f_3(x) + \dots \} = 0$$

Equating the different powers of $q$ to zero, we get the following differential equations from which to determine $f_1(x), f_2(x)$, ... ... etc.

$$\left.\begin{aligned} f_1''(x) &= -16 \cos 2nx \\ f_2''(x) &= \frac{32}{n^2} - 16 \cos 2nx \,.\, f_1(x) \\ f_3''(x) &= -16 \cos 2nx \,.\, f_2(x) + \frac{32}{n^2} f_1(x) \\ f_4''(x) &= -16 \cos 2nx \,.\, f_3(x) + \frac{32}{n^2} f_2(x) - \frac{224}{n^6} \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}\right\} \dots (3)$$

The expressions $f_1(x), f_2(x)$ ... ... ... etc. will be obtained in succession from the above relation, subject to the condition that various constants that occur in the solutions of (3) must be such as will make $f_1, f_2$ ... etc. periodic.

Thus from

$$f_1''(x) = -16 \cos 2nx,$$

we have

$$f_1(x) = \frac{4}{n^2} \cos 2nx + c_1 x + c_2,$$

Let us choose $c_1 = c_2 = 0$, then

$$f_1(x) = \frac{4}{n^2} \cos 2nx.$$

Now

$$f_2''(x) = \frac{32}{n^2} - 16 \cos 2nx \cdot \frac{4}{n^4} \cdot \cos 2nx,$$

by the above relation.

$$\therefore \quad f_2''(x) = -\frac{32}{n^2} \cos 4nx.$$

Since $f_2$ must be periodic, we have

$$f_2(x) = \frac{2}{n^4} \cos 4nx.$$

Similarly, it can be easily seen that

$$f_3(x) = \frac{4}{9n^6} \cos 6nx - \frac{28}{n^6} \cos 2nx,$$

$$f_4(x) = \frac{1}{18n^8} \cos 8nx - \frac{160}{9n^8} \cos 4nx,$$

Thus we have

$$y = c_n e_0(x, q) = 1 + \frac{q}{n} \cdot \frac{4}{n} \cdot \cos 2nx + \frac{q^2}{n^2} \cdot \frac{2}{n^2} \cos 4nx$$

$$+ \frac{q^3}{n^3} \left\{ \frac{4}{9n^3} \cos 6nx - \frac{28}{n^3} \cos 2nx \right\}$$

$$+ \frac{q^4}{n^4} \left\{ \frac{1}{18n^4} \cos 8nx - \frac{160}{9n^4} \cos 4nx \right\}$$

$$+ \quad \dots \quad \dots \quad \dots$$

$$= 1 + \left( \frac{4q}{n^2} - \frac{28q^3}{n^6} + \dots\dots \right) \cos 2x$$

$$+ \left( \frac{2q^2}{n^4} - \frac{160q^4}{n^8} + \dots \right) \cos 4x$$

$$+ \left( \frac{4q^3}{9n^6} - \dots\dots \right) \cos 6x$$

$$+ \left( \frac{q^4}{18n^8} \quad \dots\dots \right) \cos 8x$$

$$+ \quad \dots \quad \dots$$

For $n = 1$, evidently

$$c_n e_0(x, q) = c\, e_0(x, q)$$

The property of rapid convergence is maintained here also.

In conclusion, I wish to thank Prof. A. C. Banerji, and Dr. Gorakh Prasad for the interest they have taken in the preparation of this paper.

# ON AN EXPRESSION FOR A CLASS OF INTEGRAL FUNCTIONS

BY P. L. SRIVASTAVA,
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1. The following interesting lemma is due to Hardy* :—

*In order that $\phi(z)$ should be an integral function of order 1 and type $k$ $(> 0)$ in Pringsheim's terminology,† it is necessary and sufficient that $\phi(z)$ should be of the form*

$$\phi(z) = \frac{1}{2\pi i} \int_C e^{uz} d\left(\frac{1}{u}\right) du, \qquad \dots \quad (1.1)$$

*where C is a contour which includes the circle $|u| = k$, and $d(w)$ is a function regular for $|w| < \frac{1}{k}$ $\left(\text{but not for } |w| \leqslant \frac{1}{k}\right)$.*

When $k = 0$, $d(w)$ is to be an integral function of $w$ and C is to be a simple contour enclosing the origin.

Now if $\phi(z)$ is an integral function of order 1 and type $k$ $(\geqslant 0)$, then it follows from the classical work of Phragmèn and Lindelof ‡ that $\phi(z)$ satisfies the asymptotic equality

$$\limsup_{\rho \to \infty} \frac{\log |\phi(\rho e^{i\psi})|}{\rho} = \lambda(\psi), \qquad \dots \quad (1.2)$$

where $\lambda(\psi)$ is a continuous function of real variable $\psi$ for $|\psi| \leqslant \pi$, $\lambda(\pi) = \lambda(-\pi)$, $\lambda(\psi)$ does not remain negative in an interval of

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* 'On two theorems of F. Carlson and S. Wigert,' *Acta Mathematica*, Vol. 42. pp. 333—337.

† A. Pringsheim : 'Elementare Theorie der ganzen transcendenten Funktionen von endlicher Ordnung,' *Mathematische Annalen*. Vol. 58. 1904, pp. 257—342.

‡ 'Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogène dans le voisinage d'un point singulier,' *Acta Mathematica*. Vol. 31, 1908, pp. 381—406.

$\psi$ of extent $\pi$, and $-k \leqslant \lambda(\psi) \leqslant k$, the value $k$ being attained for at least one value of $\psi$. And in view of Hardy's lemma one is naturally inclined to ask whether, given any arbitrary function $\lambda(\psi)$ satisfying the conditions stated above, there exists a necessary and sufficient condition analogous to (1.1) in order that an integral function $\phi(z)$ may satisfy an asymptotic equality of the type (1 2).

It has been possible to answer this question in the affirmative only after imposing certain additional restrictions on $\lambda(\psi)$. These restrictions are that $\lambda'(\psi)$ and $\lambda''(\psi)$ are continuous except for a finite number of discontinuities of $\lambda'(\psi)$ of the first kind, such that

$$\lambda'(\psi + 0) > \lambda'(\psi - 0)^*,$$

and that $\lambda(\psi) + \lambda''(\psi) > 0$, except for a finite number of intervals of $\psi$ for which $\lambda(\psi) + \lambda''(\psi) \equiv 0$. With these limitations on $\lambda(\psi)$, the polar tangential equation

$$p = \lambda(\psi), \qquad (|\psi| \leqslant \pi), \qquad \dots (1.3)$$

represents a closed curve $\Gamma$, which may be given by one or more equations of Cartesian form. $\Gamma$ may have a finite number of angles and straight pieces on it, and, in special cases, may reduce to a point-circle, or a straight line.

Then the main theorem that I wish to establish may be stated as follows:—

THEOREM I. *Let $\Gamma$ be a closed curve given by its tangential polar equation (1·3), where $\lambda(\psi)$ satisfies the conditions stated above. Then the necessary and sufficient condition that an integral function $\phi(z)$ should satisfy the asymptotic equality*

$$\limsup_{\rho \to \infty} \frac{\log |\phi(\rho e^{i\psi})|}{\rho} = \lambda(\psi), \quad (\text{for } |\psi| \leqslant \pi), \dots (1.4)$$

*is that $\phi(z)$ should be of the form*

$$\phi(z) = \frac{1}{2\pi i} \int_C e^{zu} J(u)\, du, \qquad \dots (1.5)$$

*where $J(u)$ is an analytic function of $u$ in the region lying exterior*

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* $\lambda'(-\pi) \geqslant \lambda'(\pi)$, the sign of equality holding when $\lambda'(\psi)$ is continuous for $\psi = \pi$.

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# THE INDIAN MATHEMATICAL SOCIETY

## NOTES AND QUESTIONS.

ASSOCIATE EDITORS:

M. BHIMASENA RAO; G. A. SRINIVASAN, M.A., L.T.

AND

N. SUNDARAM IYER, M.A.

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14*th Feb.* 1929.

## Notes and Questions.

### Bhaskara and Samçlishta Kuttaka.

In a paper published in the *Bulletin of the Calcutta Mathematical Society*, Vol. XVII, (1926), pp. 89—98, Mr. Saradakanta Ganguly draws attention to a stray rule regarding the solution of the general case of Simultaneous Indeterminate Equations of the first degree, found in two palm-leaf manuscript copies of *Lilavati* with commentary in Oriya characters and also in two others of the same text in Andhra characters. In Mr. Ganguly's opinion, Bhaskara was really the author of these rules. This is however a matter of doubt. In this note we propose to re-open the question of genuineness of Bhaskara's authorship by examining the rule in the light of other Indian solutions of the same problem and also in the light of Indian mathematical tradition.

We adopt the modern notation to describe the rules and problems in question and refer the reader to Mr. Ganguly's essay for the literal rendering of the textual matter.

The rule for the solution of the general case of simultaneous indeterminate equations mentioned in the manuscripts aforesaid, is known as संश्लिष्ट बहुसामान्य कुट्टकसूत्रम् , which, for brevity, we may call Samçlishta Sutra.

It is interesting to note that the earliest Indian attempt of indeterminate analysis takes the form of solving simultaneous congruences of the type

$$x \equiv a_1 \text{ (mod. } m_1)$$

$$x \equiv a_2 \text{ (mod. } m_2)$$

$$x \equiv a_3 \text{ (mod. } m_3),$$

$$\ldots \quad \ldots \quad \ldots$$

For, though Aryabhata mentions only the case of two congruences, the rules given by him can be made to serve for more than two cases also by repeating the process. This is actually done by Mahaviracharya, (vide *Ganitasara Sangraha*, verses 127½ —129½, pp. 81-82, Text). Even if the intention of the original authors

be doubtful, the commentators make it perfectly plain that in case the congruences are more than two, the solution of the first two congruences may be combined with the third just in the same way as the first two congruences, to obtain a solution satisfying all the three congruences, this process being capable of extension to any number of them. Further, it is in accordance with Indian tradition, that authors of original scientific treatises are generally very brief, their aim being to indicate the general outline of procedure and leave the details to be worked out by the reader. To quote Bhaskara :—

उपदेशलवं शास्त्रं कुरुते धीमतो यतः ।
तत्तु प्राप्यैव विस्तारं स्वयमेवोपगच्छति ॥
जले तैलं खले गुह्यं पात्रे दानं मनागपि ।
प्राज्ञे शास्त्रं स्वयं याति विस्तारं वस्तुशक्तितः ॥

"A little instruction and guidance in science is sufficient for the intelligent student, for this alone will help him to develop his knowledge of his own accord. Science instilled into the intelligent mind has sufficient vitality in it to grow and expand by its own force even as a drop of oil on a sheet of water, a piece of secret confided to a villain, or a little act of charity to the deserving person."

The Samçlishta Sutra* which Mr. Ganguly attributes to Bhaskara amounts in effect to this:

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* The following is the text of the Sutra found in the palm-leaf manuscript (No 1885), Oriental Library, Mysore.

संश्लिष्ट बहुसामान्य कुट्टकसूत्रम् ।
हारे विभिन्ने गुणके च भिन्ने स्यादाद्य राशे र्गुणकस्तु साध्यः ।
द्वितीय भाज्यघ्न गुणः स आद्यः क्षेपोभवेत्क्षेपयुतो द्वितीयः ॥
द्वितीय भाज्यघ्न तदाद्य हारो भाज्यो भवेत्तत्र हरो हरः स्यात् ।
एवं प्रकल्प्यापि च कुट्टकेऽथ जातो गुणश्चाद्यहरेण निघ्नः ॥
गुणो भवेदाद्य गुणेन युक्तः हरघ्नहारोऽत्रहर प्रादिष्टः ।
अथ तृतीयेपि तथैव कुर्यात् एवं बहूनामपि साधयीत ॥

To solve the congruences:

$$a_1 x \equiv b_1 \pmod{m_1}$$
$$a_2 x \equiv b_2 \pmod{m_2}$$
$$\ldots \quad \ldots \quad \ldots$$

Let the solution of the first congruence be $x \equiv k_1 \pmod{m_1}$

*i.e.*, $$x = k_1 + tm_1 \quad \text{(say)}.$$

Substituting this in the second, we get

$$a_2 k_1 + a_2 tm_1 \equiv b_2 \pmod{m_2}$$

*i.e.*, $$a_2 m_1 t \equiv b_2 - a_2 k_1 \pmod{m_2}$$

where $t$ is the unknown to be determined.

Let the solution of this congruence be $t \equiv t_1 \pmod{m_2}$.

Then $$x = k_1 + m_1 t_1 \pmod{m_1 m_2}$$
$$= k_1 + m_1 t_1 + u\, m_1 m_2 \quad \text{(say)}$$

This again may be substituted in the third congruence and solved for $u$ and so on.

There is nothing striking or original in this method and it will obviously suggest itself to any one who knows how to solve the first congruence. It is a mere repetition of a process with no new principle involved and does not justify an explicit enunciation by a mathematician of the attainments of Bhaskara. While Aryabhata, Brahmagupta and Mahaviracharya had thought it fit to suggest similar obvious extensions of their method as being implied in the particular cases discussed by them, it is at least strange that Bhaskara who is content with उपदेशलव should deem it proper to dwell on it at length. If he should have really done so, he has gone out of the way to play the role of the prolix commentator. It is, therefore, very likely that a commentator has interpolated it in Bhaskara's text.

Mr. Ganguly suggests that this samçlishta rule must have been felt difficult in those days, and as it had no application in astronomy, readers might not be inclined to waste their time in learning and copying the rule, etc. In my opinion, the omission is not due to the difficulty for there is none

in it, but may be due to its being too obvious for explicit mention. In this conection it must be noticed that other Hindu mathematicians have not neglected the problem on account of its difficulty but have shown their full appreciation by suggesting alternative methods of solution. Aryabhata was the first to hint the general problem of finding a number having given residues with respect to given moduli. Mahaviracharya who lived more than 250 years before Bhaskara has given a rule different from the samçlishta rule. He solves the congruences $a_1 x \equiv b_1$ (mod, $m_1$), $a_2 x \equiv b_2$ (mod. $m_2$), etc. separately and obtains $x \equiv k_1$ (mod $m_1$), $x \equiv k_2$ (mod $m_2$)... etc. which he manipulates in Aryabhata's fashion to get the final result. This is certainly more original and natural than the samçlishta rule and is also the accepted modern method. Bhaskara indeed goes further and suggests a novel method of solving simultaneous equations in the apparently particular case of the same modulus for all congruences. There is no loss of generality in this uniform modulus, for we can replace the given set of congruences by the equivalent set

$$\frac{m}{m_1} a_1 x \equiv \frac{m}{m_1} b_1 \text{ (mod } m), \quad \frac{m}{m_2} a_2 x = \frac{m}{m_2} b_2 \text{ (mod } m) \ \ldots\ldots$$

where $m$ is the L. C M. of $m_1$ $m_2$, ... We shall deal with this aspect of Bhaskara's Samçlishta Kuttaka* in some detail presently. Again, one of the commentators of Aryabhata in a work Kuttakara Çiromani† explicitly devoted to Kuttaka problems distinguishes three varieties:

(1) that which has a single modulus but different residues;

(2) that in which there is only one dividend co-efficient but different divisors as well as different residues;

(3) that in which the dividend co-efficients, the divisors and residues are all different.

The rule given for the last case is just the same as Mahaviracharya's.

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* Samçlishta kuttaka is different from the spurious samçlishta rule.

† This is a work of one Devaraja; it is meant to be a commentary and supplement to Aryabhata's Kuttaka Sutram. Four manuscript copies of this work are available in the Oriental Library, Mysore.

Brahmagupta also suggests the same method. Thus, nowhere does the spurious samçlishta rule seem to be recognised; at any rate, even if recognised, probably it is not considered more worthy of adoption than the other one.

It is wrong to argue that the omission of the samçlishta rule in many of the texts of *Lilavati* and *Bijaganita* as well as the absence of reference to it in other Hindu mathematical texts, is due to its difficulty. For more difficult rules such as the Chakravala or the cyclic method of solving indeterminate equations of the second degree have not been omitted but found their due place among all the known manuscript copies of *Siddhanta Çiromani*. As regards the application to astronomy, it is not difficult to devise one, when applications have been found for even the Chakravala in Indian astronomy. In fact Brahmagupta gives an important astronomical application, *viz.*, 'Given the residual revolutions of different planets, to find the elapsed number of days.'

So much for the spurious rule attributed to Bhaskara. Apart from this, there is a well-known rule for Samçlishta Kuttaka, the importance and significance of which have not been sufficiently recognised. According to Mr. Ganguly, the wording of this well-known rule strongly suggests that Bhaskara must have considered the case of the spurious rule, without which the chapter would remain incomplete. The invalidity of this argument can be felt forcibly when applied to a similar rule * given by the younger Aryabhata without any discussion of the case of different divisors, corresponding to the spurious rule. Unlike the spurious rule, this well-known rule is very popular and copied by Bhaskara's successors. There is something deeper in the rule than what appears at first sight. It furnishes a novel method of solving simultaneous congruence equations, with a common modulus. As we have already pointed out, the common modulus does not in any way make the method lose its generality. But a note of warning is necessary. Bhaskara has not done him-

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* गुणकैक्यं संश्लिष्टे भाज्यः शेषैक्यकं भवेत् क्षेपः ।
तुल्यच्छेदे कर्म मन्दार्थे कथ्यते विततः ॥

*Maha Siddhanta* of Aryabhata.

This is just a paraphrase of Bhaskara's एको हरश्चेत्........

self justice in this rule, for he does not point out the cases of failure though it is his usual custom † to point out खिल cases.

The point is this: If we have a system of congruences

$$\left.\begin{array}{l} a_1 x \equiv b_1 \\ a_2 x \equiv b_2 \\ a_3 x \equiv b_3 \\ \dots \quad \dots \end{array}\right\} \text{(mod. } m) \quad \dots \qquad \dots \quad (\alpha)$$

any linear combination

$$\Sigma\, l_r\, a_r\, x \equiv \Sigma\, l_r\, b_r \text{ (mod. } m) \quad \dots \qquad \dots \quad (\beta)$$

gives the value of $x$ which satisfies all the given congruences, provided $\Sigma\, l_r\, a_r$ is prime to $m$ and the congruences are consistent. While Bhaskara has recognised the principle of the linear combination, he has failed to note the limitation about $\Sigma\, l_r a_r$ as well as the consistence of the congruences. Bhaskara's method consists in using unity for each of $l_1, l_2, \dots$ and solving the representative congruence $\Sigma\, a_r\, x \equiv \Sigma\, b_r$ (mod. $m$). Bhaskara's rule is particularly serviceable in the congruences

$$a_1 x \equiv b_1 \bmod (m_1),\ a_2\, x \equiv b_2 \bmod (m_2), \dots \dots$$

where $m_1, m_2, \dots$ are prime to each other and $m_1$ is prime to $a_1$, $m_2$ to $a_2$ and so on. Thus, taking the illustration for the spurious rule,

$$7x \equiv 3 \text{ (mod. } 62);\ 6x \equiv 5 \text{ (mod. } 101);\ 8x \equiv 9 \text{ (mod. } 17)$$

we may combine them into a single representative congruence

$$(7 \times 17 \times 101 + 6 \times 17 \times 62 + 8 \times 62 \times 101)\, x$$
$$\equiv 3 \times 17 \times 101 + 5 \times 17 \times 62 + 9 \times 62 \times 101$$
$$\text{(mod. } 62 \times 101 \times 17)$$

*i.e.* $68439x \equiv 66779 \text{ (mod. } 106454)$

which yields $x \equiv 70583 \text{ (mod. } 106454)$.

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† cf: येनच्छिन्नौ भाज्य हारौ न तेन क्षेपश्चैतद्दुष्टमुद्दिष्टमेव ।

or again

रूपशुद्धौ खिलोद्दिष्टं वर्गयोगो गुणोनचेत् ।

This shows again how this rule about Samclishta Kuttaka covers also the case of the spurious rule and the completeness of the chapter does not suffer by the omission of the latter.

We may conveniently suppose, without loss of generality, that the H.C.F. of $a_1, a_2, \ldots$ in the congruences ($\alpha$) is prime to $m$, for if there be a common factor between the H.C.F. and $m$, the whole congruence may be divided out by this common factor. If the congruences are consistent, there is only one value of $x$ less than $m$ which satisfies all of them. If $\Sigma\ l_r a_r$ and $m$ have the highest common factor, say $d$, then there are $m/d$ values ($< m$) for the congruence ($\beta$), only one of which is bound to satisfy ($\alpha$) and we have to pick out this value by trial.

It is important to note that the least value of ($\beta$) need not always satisfy ($\alpha$) except when $\Sigma\ l_r a_r$ is prime to $m$. In Bhaskara's example, the least value of the representative congruence $15x \equiv 21 \pmod{63}$ or $5x \equiv 7 \pmod{21}$ seems to satisfy the given congruences

$$\left.\begin{aligned} 5x &\equiv 7 \\ 10x &\equiv 14 \end{aligned}\right\} \pmod{63}.$$

which is a matter of chance and has perhaps led to Bhaskara's oversight in assuming probably that every solution of the new congruence is the required one.

If the given congruences be

$$\left.\begin{aligned} 5x &\equiv 7 \\ 30x &\equiv 42 \end{aligned}\right\} \pmod{63}.$$

Bhaskara's method will give $35x \equiv 49 \pmod{63}$

or $5x \equiv 7 \pmod{9}$ as the representative congruence with $x \equiv 5$ as the least value. This evidently does not satisfy the given congruences. We have to choose the next higher value 14, which suits them. Gaṅéśa in his commentary *Buddhivilasini* of *Lilavati* has struck this note of warning: The quotient as it comes out in this operation is not to be taken; but it is to be separately sought with the several original multipliers applied to this quantity and divided by the divisor as given.

A. A. KRISHNASWAMI AYYANGAR.

## Generalisation of Some Properties of the Pedal Line.

1. Let a transversal cut the sides BC, CA, AB of a triangle ABC at L, M and N. If the perpendiculars at L, M, N to the sides on which they lie are concurrent at a point X, then we have the following well-known theorems of elementary geometry :—

(i) X lies on the circum-circle of ABC;

(ii) LMN bisects the line joining X to the ortho-centre of ABC.

(iii) LMN envelopes a tricusp hypocycloid.

What happens if the perpendiculars at L, M, N to the sides of ABC are *not* concurrent? The answer is contained in the Theorems II, III and IV below. Theorem II* was published by me in the *Mathematical Gazette*, Vol. XI, p. 20.

2. As a preliminary, let us consider some properties of directly similar triangles.

If ABC and A′B′C′ be two directly similar triangles, then by turning one of them in its own plane through a suitable angle ($\theta$), it can be made homothetic with the other. For every point X belonging to ABC, there is a definite *corresponding* point X′ belonging to A′B′C′. In general X and X′ are different, but in one case they are coincident; this is the *double point* or the *centre of similitude* O of the triangles. The ratio OX : OX′ is constant (= AB : A′B′) and the angle XOX′ is also constant whatever pair of corresponding points X and X′ we may take. Further the angle between every pair of corresponding lines is also $\theta$: in particular each side of ABC makes the angle $\theta$ with the corresponding side of A′B′C′. If the two triangles are homothetic, O is the point of concurrence of AA′, BB′, CC′.

Given ABC and A′B′C′, O is obtained as the second point of intersection of the circles BB′L and CC′L where L is the point where BC and B′C′ meet. It is easily seen that if CA, C′A′ cut at M and AB, A′B′ at N, then OAA′MN, OBB′NL, OCC′LM are cyclic.

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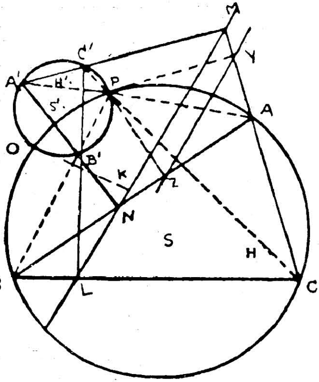
* Since writing out this paper, I find that Theorem III is proved, by a different method, by N. M. Gibbins in *Math. Gaz.*, Vol. XII, p. 440, and that an incorrect form of Theorem IV is stated by L. J. Rogers (*Math. Gaz.*, Vol XIV, p. 142).

Again, if BB′ and CC′ cut at P, then OBCP, OB′C′P are cyclic. If AA′ also passes through P, we see that OBAP, OB′A′P are also cyclic. Hence we have the following :

THEOREM I. *If two directly similar triangles are in perspective, their centres of similitude and of perspective are the points of intersection of their circum-circles.*

This theorem ceases to be true when O and P coincide, *i.e.*, when the triangles are homothetic.

3 THEOREM II. *If a transversal cuts the sides BC, CA, AB of a triangle ABC at L, M, N and if B′C′L, C′A′M, A′B′N are drawn perpendicular to BC, CA and AB respectively, then the circum-circles of ABC and A′B′C′ cut orthogonally.*




For the triangles ABC and A′B′C′ are directly similar and in perspective, LMN being the axis of perspective.

Hence if P is their centre of perspective and O their centre of similitude, O and P are the points of intersection of their circum-circles. Now the angle between any pair of corresponding lines in the two triangles must be a right angle. Hence the angle between OS and OS′ is

90°, where S, S′ are the circum-centres of ABC and A′B′C′. Hence the two circum-circles cut orthogonally.

If the perpendiculars at L, M, N are concurrent at X, then the points A′, B′, C′ coelesce at X and the circle A′B′C′ becomes the point circle at X. Hence X lies on the circle ABC—which is the theorem of § 1 (i).

4. If PY, PZ are drawn perpendicular to CA and AB, then since A, P, A′ are collinear, YZ is parallel to MN, that is, *the pedal line of P with respect to the triangle ABC is parallel to LMN* * Similarly the pedal line of P with respect to A′B′C′ is also parallel to LMN.

5 THEOREM III. *LMN bisects the line joining the ortho-centres of triangles ABC and A′B′C′.*

Let H and H′ be the ortho-centres. Let Σ be the directrix of the parabola having its focus at P and touching the sides of ABC; let Σ′ be the directrix of the parabola having its focus at P and touching the sides of A′B′C′. Since by § 4, the pedal lines of P with respect to the two triangles are parallel, these two parabolas are confocal and Σ, Σ′ are both parallel to LMN. It is a well-known property of confocal parabolas that the locus of the point of intersection of two perpendicular tangents, one to each parabola, is a straight line parallel to and midway between their directrices. Now, L, M, N are the points of intersection of perpendicular tangents to two confocal parabolas. Hence LMN is midway between Σ and Σ′ But H lies on Σ and H′ on Σ′. Therefore HH′ is bisected by LMN.

6 A further generalization of Theorem III and the corresponding theorem in § 1 (ii) may be attempted by relaxing the condition that the sides of A′B′C′ should be perpendicular to the corresponding sides of ABC. The following is a first step in this process and may be proved by cylindrical projection :—

*If two triangles ABC and A′B′C′ are in perspective, LMN being their axis of perspective, and if the lines through A, B, C parallel to B′C′, C′A′, A′C′ are concurrent at a point H, then the lines through A′, B′, C′ parallel to BC, CA, AB are also concurrent at a point H′ and LMN bisects HH′.*

* *Vide* also *Mathematical Gazette*, Vol. XI, p. 87 : Note by E. P. Lewis.

7. From § 2 it follows that O is the point of intersection of the circum-circles of the four triangles formed by the lines BC, CA, AB and LMN, taken three by three. So the feet of the perpendiculars from O to these four lines are collinear; in other words, the pedal line of O with respect to ABC passes through K, the foot of the perpendicular from O on LMN. Similarly the pedal line of O with respect to A′B′C′ also passes through K: in fact the two pedal lines cut at right angles at K.

Given triangle ABC and tranversal LMN, the triangle A′B′C′ becomes definite and we see that the point O is the unique focus of the parabola touching LMN and the sides of ABC, while P is the unique point whose pedal line with respect to ABC is parallel to LMN.

On the other hand, if two points O and P on the circum-circle of ABC are given, the circle through O and P cutting circle ABC orthogonally is unique but there are two triangles A′B′C′ possible, *viz.*, one which has, with ABC, O for centre of similitude, P for centre of perspective and the other having P for centre of similitude and O for centre of perspective. Thus, while for each LMN, only one line OP is got, for each OP there correspond two lines LMN. Or the correspondence between OP and LMN is (1, 2). If O and P coincide, the two LMN's coincide with the pedal line of O.

8. THEOREM IV. *If the circle A′B′C′ is of constant size, then the envelope of LMN consists of two tricusp-hypocycloids.*

The two hypocycloids are equal in size and both touch the sides of the triangle ABC. When the line LMN becomes a pedal line of the triangle, the two tricusps coelesce with the Steiner tricusp. These hypocycloids are studied in detail by Converse in his paper on "A system of hypocycloids" in the *Annals of Mathematics*, Series II, Vol. 5, (1904), pp. 105—139.

L. J. Rogers in his note in the *Mathematical Gazette*, Vol. XIV, p. 142, states the result incorrectly, *viz.*, that the envelope will be a single tricusp *hypo-trochoid.*

I shall give a proof of Theorem IV in a subsequent note.

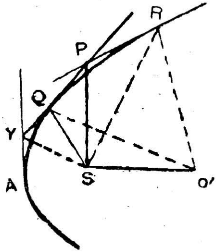
G. A. SRINIVASAN.

## Half-circle of Curvature of a Conic.

The following note contains a new method of obtaining the radius of curvature at any point of a conic. If O is the centre of the circle of curvature at any point Q on a curve, the circle on QO as diameter is referred to here as the *half-circle of curvature* at Q. If the tangents at two near points Q and R on a curve intersect at P, then it is easily seen that the limiting position of the circle PQR as R moves up to Q is the half-circle of curvature at Q.

We shall first take the case of a parabola. The following lemma will be found useful:—

> If PQ and PR are tangents from a point P to a parabola, focus S, then the circum-centre of PQR lies on the perpendicular to SP through S.




For, if O′ is the circum-centre of PQR, then the angle QO′R is twice the exterior angle at P and hence it is equal to the angle QSR. Thus Q, S, O′, R are concyclic and since QO′ = O′R, SO′ is the external bisector of ∠QSR But SP is the internal bisector of ∠QSR. Therefore SO′ is perpendicular to SP.

Let now R, and therefore P, tend to Q. Then O′ becomes the centre of the half-circle of curvature at Q and QSO′ becomes a right angle.

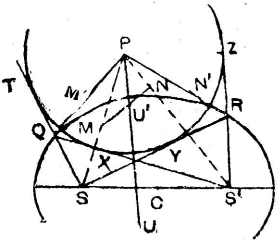
Draw SY ⊥ from S on the tangent at Q; AY is the tangent at the vertex A of the parabola. Then

$$\frac{QS}{QO'} = \frac{SY}{QS} = \frac{\sqrt{(QS.AS)}}{QS}$$

$\therefore$ $QO' = QS^{\frac{3}{2}} / AS^{\frac{1}{2}}$ = half the radius of curvature at Q.

Proceeding next to the case of a central conic, we shall require another lemma: *viz*:—

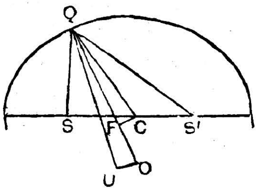
> If PQ, PR are tangents to a central conic, foci S, S′, then the common chord of the circles PQR and PSS′ is the symmedian through P of either of the triangles PQR, PSS′.




To prove this, describe the circle with centre P and touching SQ, S′Q, SR and S′R at T, X, Y an Z respectively. Let M and N be the middle-points of TY and XZ, and M′, N′ those of TX and YZ. Invert the circles PQR, PSS′ with respect to the circle XYZ. Then the circle PSS′ inverts into the straight line MN and the circle PQR inverts into the line M′N′. Hence U′ the point of intersection of MN and M′N′ is the inverse of U the other point common to the circles PQR and PSS′ and PU′U is their common chord. But we know that the lines joining the mid.-points of pairs of opposite sides and the mid-points of the diagonals of a quadrilateral bisect one another. Hence U′ is the mid.-point of MN. Also M, N, S, S′ being concyclic, PU′U is the symmedian through P for the triangle PSS′.

Suppose now P tends to Q. Then QU becomes the common chord of the half-circle of curvature at Q and the circle QSS′ and is the symmedian through Q. Hence

$$QU \cdot QC = QS \cdot QS'.$$




Now let QO be the radius of curvature at Q so that QUO is a right angle; draw CF ⊥ to QO and therefore parallel to the tangent at Q. Then, QU and QC, being isogonally conjugate with respect to SQS′, are equally inclined to QO. Hence $QU \cdot QC = QO \cdot QF$.

$\therefore \quad QO = QS \cdot QS' / QF.$

$= CD^2 \cdot CD / QF \cdot CD$, where CD is the semi-diameter conjugate to CQ.

$= CD^3 / CA \cdot CB.$

S. PURUSHOTHAM.

## Solutions.

### Question 1094.

(S. R. RANGANATHAN):—If the number of A's: the number of B's: the number of C's as $a:b:c$ and if $a_1$ per cent and $a_2$ per cent of the A's are also B's and C's respectively, discuss the limits between which the percentage of B's that are also C's should lie.

*Solution by S. Audinarayanan.*

With the notation of the Theory of Attributes*, let N denote the whole universe under observation. Then we have the following five positive class-frequencies given, *viz*:

$$(A) = ka\,N, \qquad (B) = kb\,N, \qquad (C) = kc\,N.$$

$$(AB) = \frac{a_1}{100}\cdot ka\,N, \quad (AC) = \frac{a_2}{100}\,ka\,N.$$

where $k$ is the constant of proportion and is such that $ka$, $kb$, $kc$ are all positive proper fractions. The other two positive class-frequencies (BC) and (ABC) are unknown, but they must be such as to make no one of the third order frequencies negative. Writing down the values of all the third order frequencies in terms of the positive class-frequencies † and expressing the conditions that no one of them may be negative, we obtain the following eight inequalities.

$$\left.\begin{aligned}(BC) &\ngtr (B)\\ &\ngtr (C)\\ &\ngtr (B)+(AC)-(AB)\\ &\ngtr (C)+(AB)-(AC)\end{aligned}\right\} \text{ and } \begin{aligned}(BC) &\nless 0\\ &\nless (B)+(C)-N\\ &\nless (AB)+(AC)-(A)\\ &\nless (A)+(B)+(C)-N-(AB)-(AC).\end{aligned}$$

Denoting the percentage of B's that are also C's by $x$, so that $(BC) = \frac{x}{100}\,kb\,N$ and substituting the values of the other frequencies, we see that $x$ should not be greater than any of the four numbers

$$100, \ \frac{100c}{b}, \ 100 + \frac{aa_2}{b} - \frac{aa_1}{b}, \ \frac{100c}{b} + \frac{aa_1}{b} - \frac{aa_2}{b} \qquad \dots (1)$$

and should not be less than any of the four numbers

$$\left.\begin{aligned}&0, \ 100 + \frac{100c}{b} - \frac{100}{kb}, \ \frac{aa_1}{b} + \frac{aa_2}{b} - \frac{100a}{b},\\ &\qquad 100 + \frac{100a}{b} + \frac{100c}{b} - \frac{100}{kb} - \frac{aa_1}{b} - \frac{aa_2}{b}.\end{aligned}\right\} \dots (2)$$

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* *cf.* Udny Yule : ***Introduction to the Theory of Statistics.*** (1919) Ch. I.

† *Ibid* : Ch. II, p. 21.

Thus $x$ should lie between the least of the first four numbers (1) and the greatest of the four numbers (2).

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## Question 1411.

(K. J. SANJANA, M.A.):—(1) The bisector of the angle A of a triangle ABC meets BC in K; Q is the mid.-point of BC and X, R are respectively the feet of the perpendiculars on it from the in-centre I and the vertex A. Prove that QX is the mean proportional between QK and QR.

(2) The triangle PQR is right-angled at R and any point K is taken in QR; QX is taken along QR equal to the mean proportional between QK and QR, and the straight line through K parallel to the bisector of the angle P meets the perpendicular to QR at X in I. If N is the mid.-point of PQ, prove that $NI = NQ \mp IX$. (The lower sign is to be taken when X lies in RQ produced.)

(3) Employ (1) and (2) to establish the tangency of the nine-point circle of a triangle with the in-circle and ex-circles of the triangle.

*Solution by Professors T. P. Trivedi and K. J. Sanjana.*
*Also solved by T. S. Venkataraman and M. V. Seshadri.*

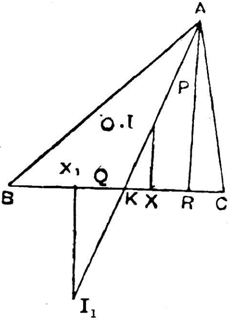


FIG. 1.

Let $I_1$ be the ex-centre and $I_1X_1$ the perpendicular from $I_1$ to BC.

It is easily seen that the range $(AK, II_1)$ is harmonic: so the range $(RK, XX_1)$ is harmonic. Also $BQ = CQ$ and $BX_1 = CX$ (each being $= s - c$); hence Q is the mid. point of $XX_1$.

Hence by the harmonic property QX is a mean proportional between QK and QR.

(2) Draw PS bisecting $\angle QPR$, to meet QR. Let M be the mid.-point of QR so that MN parallel to PR is perpendicular to QR. Draw NH parallel to QR to meet IX.

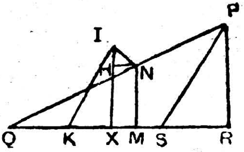


FIG. 2.

$$\text{Now } \frac{QX}{QR} = \frac{QK}{QX} (\text{data}) = \frac{QX - QK}{QR - QX} = \frac{KX}{XR}$$

$$\therefore \quad QR . KX = QX . XR = QM^2 - XM^2 = QM^2 - HN^2$$

$$\therefore \quad HN^2 = QM^2 - 2\, QM . KX \quad \ldots \quad \ldots \quad (1)$$

Again from similar triangles

$$\frac{IX}{XK} = \frac{PR}{RS} = \frac{PQ}{QS} = \frac{PR + PQ}{QR} = \frac{NM + NQ}{QM}$$

$$= \frac{QM\,(NM + NQ)}{NQ^2 - NM^2} = \frac{QM}{NQ - NM}$$

$$\therefore \quad IX . NQ - IX . HX = QM . KX \quad \ldots \quad \ldots \quad (2)$$

$$\text{Now } IN^2 = HN^2 + IH^2 = QM^2 - 2\,QM . KX + (IX - HX)^2 \text{ from (1)}$$

$$= QN^2 - HX^2 - 2\, IX . NQ + 2\, IX . HX + (IX - HX)^2$$

$$= (QN - IX)^2$$

$$\therefore \quad IN = QN \sim IX.$$

Similarly by taking X in RQ produced, we can prove that

$$IN = NQ + IX.$$

(3) In Fig (1) let QP be the diameter of the nine-point circle through Q so that N is its mid.-point. If O be the circum-centre, the line AIK bisects $\angle OAR$. As OA is parallel to PQ. IK is parallel to PS, the bisector of $\angle QPR$. Now from Fig. (2) with the same notation as in Fig. (1), $IN = QN - IX = \frac{1}{2} R - r$, so that the nine-point circle touches the in-circle. Similarly taking X in RQ produced, we can show that the nine-point circle touches the ex-circle opposite the angle A.

[Prof. V. A. Apte points out that this question is completely solved in Sanjana's *First year's course in Mathematics* (1914 Edn.), Miscellaneous Examples VI, Qn. 19 and Ex. 419. The following is a list of other proofs of Feuerbach's Theorem that have appeared in our *Journal*:

| | | | |
|---|---|---|---|
| Progress Report | 7 (1908), | p. | 10 (M. T. Naraniengar), |
| Volume | 3 (1911), | ,, | 193 (R. P. Paranjpye) |
| ,, | 9 (1917), | ,, | 94 (J. H. Grace) |
| ,, | ,, | ,, | 228 (T. Krishna Rao) |
| ,, | 15 (1923-4) | ,, | 52 (K. J. Sanjana).—*Ed.*] |

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## Question 1501.

(K. J. SANJANA, M.A.):—

(i) Prove that one solution of the equation $x^2 - 67y^2 = 1$ is

$$x = 48842, \ y = 5967,$$

and show that successive values of $x$ and $y$ forming a definite series can be found to satisfy the equation.

(ii) Prove that one solution of the equation $x^2 - 61y^2 = 1$ is

$$x = 1766, 319, 049, \ y = 226, 153, 980,.$$

and show how other solutions may be obtained successively.

[These problems have been taken from the *Kuttaka*,—Pounding or Pulverising section—of Bhaskara's *Bija Ganita*. I give below the *Sloka* in which he has propounded the question:—

*Ká saptashashthigunitá kritirekayuktá*
*Ká chaikashashthinihatá cha sakhe sarúpá* |
*Syánmúladá yadi kritiprakritir—nitantam*
*Tvachchetasi pravada táta tatá latávat* || ].

*Solution and Remarks by the Proposer.*

*Also solved by I. J. Chowla, N. Gagannadha Rao and N. Maitra.*

§ 1. (i) Reducing $\sqrt{67}$ to a continued fraction in the ordinary way, we have the denominators 1, $\overset{*}{3}$, 6, 7, 9, 2, 9, 7, 6, 3, $\overset{*}{1}$, ..., corresponding to the partial quotients

$$8, \overset{*}{5}, 2, 1, 1, 7, 1, 1, 2, 5, \overset{*}{16}, \ldots$$

Forming the successive convergents we get for the tenth convergent, pertaining to the quotient 5 preceding 16, the fraction 48842 / 5967. As

the tenth convergent is greater than $\sqrt{67}$ its terms make the right-hand side $+1$; thus Bhaskara's solution is obtained.

(ii) Proceeding similarly with $\sqrt{61}$, we get the partial quotients

$$7, \dot{1}, 4, 3, 1, 2, 2, 1, 3, 4, 1, \dot{14}, \ldots \ldots$$

Forming the successive convergents we get for the 11th, pertaining to the quotient 1 preceeding 14, the fraction 29718/3805. As the 11th convergent is less than $\sqrt{61}$, its terms make the right-hand side negative: thus we obtain

$$29718^2 - 61 \times 3805^2 = -1 \ldots \quad \ldots \quad \ldots \quad (1)$$

Bhaskara now finds, by means of rules previously laid down in the theoretical part of the *Kuttaka*, the following two solutions—

$$x = \frac{39}{2}, \; y = \frac{5}{2} \text{ for } -1 \text{ and } x = \frac{1523}{2}, \; y = \frac{195}{2} \text{ for } +1$$

From these by combination of like sets he at once deduces, by a method called *Samâsa* Bhâvanâ, the integral roots for positive unity exactly as given in the question. This interlinking process (*bhávaná*) is in fact the same as the method for obtaining any number of solutions of the equation $x^2 - Ny^2 = p$ when one solution in integers has been obtained. (See Hall and Knight, *Higher Algebra*, Chap. 28, §§ 375-6). The reader's attention is sure to be arrested by the great utility and modernness of this method. This process is technically called *Bhávaná*; it is said to be *samása* (composition) *Bhávaná* when the plus sign and *antara* (difference) *Bhávaná* when the minus sign is used between the two derived products.

We can also proceed by employing the first part of the following theorem of mine contained in Question 57 (*J.I.M.S.*, Nov. 1908 and June 1909).

"Let N be a rational number greater than unity and $N^{\frac{1}{2}}$ when reduced to a continued fraction. have a recurring period of $k$ quotients; denote the $k$th, $2k$th, ... $nk$th convergents by

$$p_1/q_1, \; p_2/q_2, \; \ldots \; p_n/q_n;$$

show that

$$p_n/q_n = (N\, q_{n-1}/p_{n-1} + p_1/q_1) \div \{(p_1/q_1).(q_{n-1}/p_{n-1}) + 1\},$$

and that when these fractions are put in their simplest forms the series for $p$ and $q$ are recurring, both with the same scale of relation."

Putting $n = 2$, we have $p_2/q_2 = (N\, q_1/p_1 + p_1/q_1) \div$

$$\{(p_1/q_1).(q_1/p_1) + 1\} = (N\, q_1^2 + p_1^2)/2\, p_1 q_1.$$

Hence in the present case

$$p_2/q_2 = (61 \times 3805^2 + 29718^2)/2(29718)(3805)$$
$$= (2 \times 29718^2 + 1)/(29718)(7610), \text{ from equation (1),}$$
$$= (1766,319,049)/(223,153,980),$$

the numerator and denominator of which fraction give the required values of $x$ and $y$.

Further solutions may be found in both cases by the use of this formula. In the case of $\sqrt{67}$, $p_2/q_2$, $p_3/q_3$, ..., and in that of $\sqrt{61}$, $p_4/q_4$, $p_6/q_6$, ..., give other corresponding values of $x$ and $y$. These values as well as the common scales of relation of the $p$'s can be found by means of the recurring series obtained in the solution of Question 57 stated above.

§ 2. Bhaskara has given the full theory and detailed method of solving indeterminate equations of the second degree. I quote here H T. Colebrooke's* accurate translation of Bhaskara's prose solution of the equation $x^2 - 67y^2 = 1$. The reader should note that C, 67, is the multiplier of $y^2$; L (*laghu*) is the lesser root $y$, G (*guru*) is the greater root $x$; and A (the additive) is the rational denominator in the process of conversion. Bhaskara here refers to the well-known theorem that any such rational denominator multiplied by $(-1)^n$, where $n$ is its number, must be the additive for the solution of such an equation to be possible.

"What is the square, which, being multiplied by sixty-seven, and one being added to the product, will yield a square root?; and what is that, which multiplied by sixty-one,† with unity added to the product, will do so likewise? Declare it, friend, if the method of the affected square be thoroughly spread, like a creeper, over thy mind."

Statement of Example 1st.: (Putting unity for the "least" root, and negative three for the additive.) C 67 L 1 G 8 A $\dot{3}$.

Making the "least" root the dividend, the "greatest" root the additive, and the additive the divisor, the statement for the operation of finding the multiplier (Ch. 2) is $\frac{\text{Dividend } 1}{\text{Divisor } \dot{3}}$ Additive 8. Here by the rule § 61, the series is $\begin{matrix}0\\2\\0\end{matrix}$; and the quotient and multiplier are found $\begin{matrix}0\\2\end{matrix}$; which, as the

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* (1) Henry Thomas Colebrooke F.R.S.. *M. Linn and Geol. Soc.* etc. ; "*Algebra (with Arithmetic and Mensuration) from the Sanscrit of Brahmegupta and Bháscará*" (London, John Murray, 1817).

† I do not give the translation of the method for the multiplier 61.

number of quotients. [in the series] is uneven, must be subtracted from the abraders (§ 57) leaving $\frac{1}{1}$; and the quotient obtained in the abrading of the additive is to be added (§ 61) to the quotient here found; making the quotient and multiplier $\frac{3}{1}$ since the divisor is negative, the quotient is considered so too (§ 68); and the quotient and multiplier are $\frac{\dot{3}}{1}$. Then the square of the multiplier 1, being subtracted from the given co-efficient 67, leaves 66; which, however, is not a small remainder. Putting therefore negative two for the assumed number by § 61, and multiplying by that negative divisor $\dot{3}$, and adding the product to the multiplier, a new multiplier is found, *viz.* 7. Its square 49 being subtracted from the co-efficient 67, the remainder 18, divided by the original additive $\dot{3}$, yields $\dot{6}$; the sign of which is reversed, as the subtraction was of the square of the multiplier from the co-efficient; and it thus becomes positive. The quotient answering to the multiplier, *viz.* $\dot{5}$, is the 'least' root. Whether this be negative or affirmative, makes no difference in the further operation. It is noted then as 5 positive. Its square being multiplied by the co-efficient, and six being added to the proluct, and the square root being extracted, the "greater" root comes out 41. Statements of these again for a further investigation of a pulverizer $\frac{\text{Dividend } 5}{\text{Divisor } 6}$ Additive 41. Here the multiplier is found, 5. Its square, subtracted from the co-efficient, leaves 42; which, divided by the original additive 6, yields 7; the sign whereof is reversed because the subtraction was from the co-efficient; and the new additive comes out $\dot{7}$. The quotient answering to the multiplier is the "least" root, 11. Hence the "greatest" root is deduced, 90.

Statement of these again for a further pulverizer:

Divid. 11<br>Div. $\dot{7}$ Add 90.

By the rule § 61, the abraded additive becomes 6 and the multiplier is found 5; and, since the products in the series are uneven, it is subtracted from its abrader, and the multiplier becomes 2. Its negative divisor (the former additive) being negative ($\dot{7}$) is multiplied by negative one ($\dot{1}$) assumed by § 64 and added to that multiplier, for a new multiplier 9; from the square of which 81, subtracting the given co-efficient 67 the remainder 14, divided by the original additive $\dot{7}$, gives the new

additive $\dot{2}$. The quotient answering to the multiplier is the "least" root 27; whence the "greatest" root is found 221. From these, others are to be deduced by combination of like sets.

Statement: L 27 G 221 A $\dot{2}$
$l$ 27 $g$ 221 $a$ $\dot{2}$

Proceeding as directed, the roots are found L 11934 G 97684 A 4. These roots divided by the root of the additive four, *viz.* 2, give roots which answer to additive unity: L 5967 G 48842 A1."

The solution is lengthy simply from the fact that Bhaskara gives the whole process of converting $\sqrt{67}$ into a continued fraction, quotes chapter and verse for each and every operation and writes down all the intermediate equations. If only the chief steps and the resulting equations of the given type are written down, we get a table of the following form :—

| Step No. | Denr. | Par. Quot. | $p_n = G$ | $q_n = L$ | $(-1)^n A_n = p^2_n - 67q^2_n$ |
|---|---|---|---|---|---|
| 1 | 1 | 8 | 8 | 1 | − 3 |
| 2 | 3 | 5 | 41 | 5 | + 6 |
| 3 | 6 | 2 | 90 | 11 | − 7 |
| 4 | 7 | 1 | 131 | 16 | + 9 |
| 5 | 9 | 1 | 221 | 27 | − 2 |
| 6 | 2 | 7 | 1678 | 205 | + 9 |
| 7 | 9 | 1 | 1899 | 232 | − 7 |
| 8 | 7 | 1 | 3577 | 437 | + 6 |
| 9 | 6 | 2 | 9053 | 1106 | − 3 |
| 10 | 3 | 5 | 48842 | 5967 | + 1 |

The reader will perceive that this process tallies exactly with the process for $x^2 - 13y^2$ given in tabular form in Chrystal's *Algebra* (first edition), Vol. 2, Ch. 33, § 17.

## Questions for Solution.

*Proposers of Questions are requested to send their own solutions along with their questions,*

**1532.** (V. RAMASWAMI AIYAR, M.A.):—Let the tangents at points P, Q of a given conic X intersect in T.

(*a*) Considering the in- and ex- centres of the triangle TPQ, show that if *one* of them lies on the curve, then, *another* of them will also lie on the curve.

(*b*) Show that the locus of T in order that such may be the case is a confocal conic, Y.

(*c*) Show that the relation between the conics X and Y, in this respect, is reciprocal.

(*d*) If X is a parabola, then Y is the confocal parabola having identical latus-rectum.

**1533.** (V. RAMASWAMI AIYAR, M.A.):—Let $a, b, c, \ldots k$ be $n$ positive quantities. Let $P_r$, $p_r$ denote the sum, and the number, of their $r$-ary products; and $H_r$, $h_r$ the sum and the number, of their $r$-ary "homogenous products."

If the quantities be not all equal, show that

(*a*) $$\frac{P_1/p_1}{1} > \frac{P_2/p_2}{P_1/p_1} > \frac{P_3/p_3}{P_2/p_2} > \ldots > \frac{P_n/p_n}{P_{n-1}/p_{n-1}};\text{ and}$$

(*b*) $$\frac{H_1/h_1}{1} < \frac{H_2/h_2}{H_1/h_1} < \frac{H_3/h_3}{H_2/h_2} < \ldots \text{ (without limit).}$$

Hence. or otherwise, show that

(*c*) $$\frac{P_1}{p_1} > \left(\frac{P_2}{p_2}\right)^{\frac{1}{2}} > \left(\frac{P_3}{p_3}\right)^{\frac{1}{3}} > \ldots > \left(\frac{P_n}{p_n}\right)^{\frac{1}{n}};\text{ and}$$

(*d*) $$\frac{H_1}{h_1} < \left(\frac{H_2}{h_2}\right)^{\frac{1}{2}} < \left(\frac{H_3}{h_3}\right)^{\frac{1}{3}} < \ldots \text{ (without limit).}$$

Hence, or otherwise. show that

(*e*) when $P_r$ is given, $P_1, P_2, \dots P_{r-1}$ have their minimum values, and $P_{r+1}, \dots P_n$ have their maximum values, when the $n$ quantities are equal; and

(*f*) when $H_r$ is given $H_1, H_2, \dots H_{r-1}$ have their maximum values, and $H_{r+1}, H_{r+2}, \dots$ (without limit) have their minimum values, when the $n$ quantities are all equal.

**1534.** (K. SATYANARAYANA) :—Prove that in all triangles in the non-Euclidean plane in which the base angles are supplementary, the median bisecting the base has the same constant length.

**1535.** (M. G. INAMDAR) :—Find all rational triangles, one of whose angles is 60°.

**1536.** (C. N. SRINIVASIENGAR) :—Prove that

(1) The Hessian of a skew ruled cubic surface is the square of a quadric.

(2) The parabolic curve of any skew ruled surface consists of a number of isolated generators.

**1537.** (R. VAIDYANATHASWAMY) :—If $l(M, N)$, $g(M, N)$ denote the least common multiple and the greatest common divisor of M, N, show that

$$\sum \left\{ l\left(\frac{M}{\delta_1}, \frac{N}{\delta_2}\right) \right\}^a \left\{ g(\delta_1, \delta_2) \right\}^{2a} = \sum \left(\frac{MN}{\delta_1 \delta_2}\right)^a \left\{ (g\,\delta_1, \delta_2) \right\};$$

the summations extending over all divisors $\delta_1$ of M and $\delta_2$ of N.

## UNSOLVED QUESTIONS FOR WHICH SOLUTIONS ARE INVITED.

**137.** (V. R. SONTI, B. A.) :—If $\sin \alpha/3$ is a root of the equation

$$2x^3 + 1 = 2x(x + 1),$$

show that the angle $\alpha$ can be geometrically trisected by means of a cardioide.

**150.** (M. T. NARANIENGAR, M.A.) :—If a rectangular lamina always immersed in a fluid rotate (in its own plane) about a fixed point in the lamina, find the locus of the centre of pressure (i) in the lamina, (ii) in space.

195331

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4 Annals of Mathematics, **29**, 4 (2 copies).
5 Astrophysical Journal, **68**, 3, 4.
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8 Journal fur die reine und angewandte Mathematik (Crelle), **159**, 4.
9 London, Edinburgh and Dublin Philosophical Magazine, **6**, 38, 39, 40.
10 Mathematical Gazette, **14**, 197 (3 copies), 198.
11 Mathematische Annalen, **100**, 3.
12 Messenger of Mathematics, **57**, 12 (2 copies).
13 Mysore University Journal, **2**, 2.
14 Philosophical Transactions of the Royal Society, **228**, 1—62.
15 Popular Astronomy, **36**, 9, 10.
16 Proceedings of the London Mathematical Society, **28**, 7 (2 copies).
17 Proceedings of the Physico-mathematical Society of Japan, **10**, 5, 6, 7, 8.
18 Proceedings of the Royal Society of London, **121**, A 787, 788 (2 copies).
19 Publications de la Faculte des Sciences de l'Universite Masaryk (1928); 98, 99, 100.
20 Rendiconti del Circolo Matematico di Palermo, **52**, 3.
21 Tohoku Mathematical Journal, **30**, 1, 2.

## Books.

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